Homogenization of a cracked saturated porous medium: theoretical aspects and numerical implementation

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Abstract

The purpose of this work is to determine, via a homogenization technique and in the framework of small strains, the macroscopic poroelastic properties of a saturated, deformable, cracked porous medium. The poroelastic matrix is assumed to be homogeneous and the cracks to be periodically distributed, the size of the period being small with respect to the size of the sample. The considered up-scaling method (based on asymptotic expansions) will provide two uncoupled mechanical and hydraulic problems describing the overall behavior of the material. The degradation of the mechanical properties due to damage is then introduced. Damage depends on cracks' opening, thus making the problem non-linear. A numerical solution of the problem is provided using finite elements. Stress-strain time-histories corresponding to an oedometric and a biaxial test are used as numerical examples. The numerical solution allows the exploration of the non-linear anisotropic behavior along with the bifurcation phenomenon.

Keywords: Asymptotic homogenization, fractured saturated porous medium, poroelasticity

1. Introduction

During the past decades, big efforts have been made to better comprehend heterogeneous materials [3, 7, 12, 25, 16]. Numerical modeling can be successfully applied to continuum problems but difficulties arise when trying to model a heterogeneous microstructure [17]. The difference between the scale of the micro and macro structures makes it difficult to determine an appropriate mesh size, leading to a computationally expensive problem if one focuses on the micro-scale, or to an approximate description of the microstructural behavior if one focuses on the macroscale problem [17].

Furthermore, macro-scale constitutive laws, calibrated with experimental results, are often adopted. This approach is however less effective when dealing with complex behaviors [11]. An alternative is provided by homogenization techniques that allow the inclusion of the micro-scale description within the macroscopic problem. In this latter framework, analytic, e.g. mixture theory Gray and Hassanizadeh [15] or semi-analytic, e.g. Eshelby [14] procedures have been developed. However, these theories can not describe the micro-macro behavior for non-linear constitutive laws or non-regular micro-structure configurations in an accurate manner (see, e.g. Kanouté et al. [16]). Numerical homogenization approaches such as direct micro-macro techniques Smit et al. [26], Miehe et al. [20], Nitka et al. [22], Nguyen [21] overcome these limitations. These techniques use numerical calculations at the (usually periodic) micro-scale level to provide a constitutive law at the macroscale. Although this approach allows more general multi-scale problems to be taken into account, it is highly computationally expensive.

The asymptotic homogenization theory documented in Bensoussan et al. [6], Sánchez-Palencia [25], Arbogast et al. [1], Papanicolau et al. [23] permits equivalent properties to be obtained and allows an analytic and a numerical approach to be combined. Based on asymptotic expansions (applied to a parameter e that relates the characteristic lengths of the two, well-separated, scales), the homogenized problem can be solved on a generic micro-structural cell (solved using, e.g. finite elements [5]) so that the homogenized macroscopic properties are finally obtained.

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This latter approach is developed herein with the purpose of determining the overall poroelastic properties of a saturated cracked deformable porous medium in the framework of small strains. We consider the deformation and the porous flow of the medium to be governed by Biot's equations of poroelasticity. The cracks are thin enough to be considered splines (surfaces in 3D). From a mechanical point of view, a crack is considered as a weakened elastic zone allowing its two lips to slip and to move apart. The relative motion of the lips induces a change of the porosity of the crack and consequently a change in the fluid flow. The opening of cracks is considered to damage the material, thus affecting the transport properties of the medium.

The first part of the paper presents the equations governing the coupled hydro-mechanical problem in the porous matrix and the cracks. The asymptotic homogenization is detailed and the final equations describing the macro-scale problem are presented. In the second part of the paper, the homogenized problem is solved numerically and some applications illustrating the effect of damage are presented, including the application of strain and stress controlled paths.

Notations.

- The "usual" vectors: positions, normal, tangent, forces, flows, ... are denoted: $\vec{x}$, $\vec{y}$, $\vec{n}$, $\vec{\tau}$, $\vec{u}$, $\vec{v}$, $\vec{T}$, $\vec{q}$, ... $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ is an orthonormal basis.
- The dot symbol $\centerdot$ denotes the simple contraction between two tensors of any order: $\vec{T} \centerdot \vec{v}$, $\vec{T} = \sigma \centerdot \vec{n}$, ...
- The colon symbol : denotes the double contraction of two second order tensors: $\sigma : \nabla\vec{v}$, $c : \epsilon\,(\vec{u})$, ...
- The tensor product $\vec{a} \otimes \vec{b}$ denotes the linear application defined by: $\forall \vec{c}$, $\left(\vec{a} \otimes \vec{b}\right) \centerdot \vec{c} = \left(\vec{b} \centerdot \vec{c}\right) \vec{a}$.
- $\overrightarrow{\mathrm{grad}}\, f$ denotes the gradient of the scalar function f, $\nabla\vec{u}$ is the gradient of the vector field $\vec{u}$ and $\epsilon\,(\vec{u})$ denotes the strain tensor associated to the displacement field $\vec{u}$, i.e. the symmetrical part $\nabla\vec{u}^S$ of $\nabla\vec{u}$. The gradients of a field of two space variables $\vec{x}$ and $\vec{y}$ are distinguished by an exponent: $\overrightarrow{\mathrm{grad}}^x f$, $\nabla^y\vec{v}$.
- Whenever the index notation of tensors is used, Einstein notation for the contraction of tensors is adopted.

2. Description of a saturated cracked deformable porous medium

Let us consider a cracked deformable and saturated porous medium occupying, in the small strain framework, a domain Ω. For the sake of simplicity the study is carried out in two dimensions; an extension to 3D is straightforward but is not presented in the following for sake of notations clarity.

The porous parts of the medium are separated by cracks which are curves that joint at points (see fig.1); Γ denotes the set of all cracks of the medium. To make the writing of the equations of the poroelasticity of the cracks precise, the cracks are (arbitrarily) oriented, let s denote the curvilinear abscissa along a crack and $\vec{\tau}$ its unit vector, assuming the crack is smooth. The unit normal $\vec{n}$ to a crack is the vector obtained by the rotation of angle $+\frac{\pi}{2}$ of the tangent vector $\vec{\tau}$.

The considered porous medium is assumed to be finely periodic. That means, on one hand, that the space distribution of cracks is periodic (see fig.1) and, on the other hand, that the size of the period is small with respect to that of the medium. In the asymptotic expansion method of homogenization used in this paper, the ratio of the size of the period to that of the medium is a small parameter intended to go to 0. That means that the periodic cells of the medium are increasingly smaller. The usual way to handle this is to define the cells of the medium as the image of a given cell Y by a homothety of ratio e, e being the small parameter of the asymptotic procedur (see Bensoussan et al. [6] and Sánchez-Palencia [25]).

A function defined on Y is said to be Y-periodic if it takes equal values on opposite sides of the cell Y.

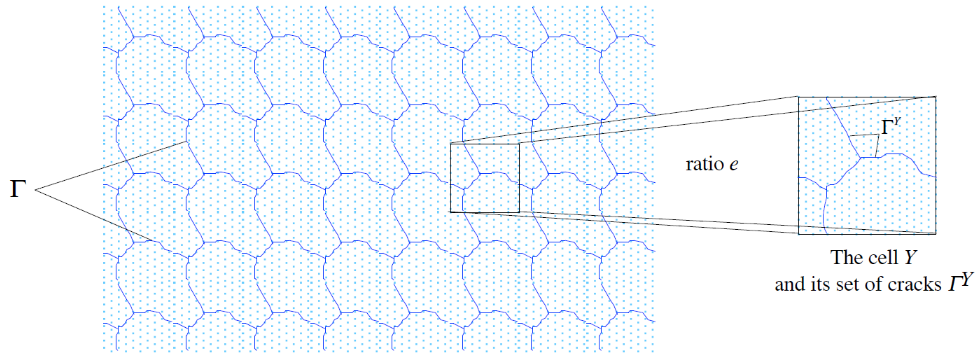

Figure 1: Periodic cracked porous medium.

Biot's equations of the porous parts. In the porous parts of the medium Ω, the deformation of the medium and the flow of fluid are governed by Biot's equations that read, see (see Biot [7, 8], Auriault [4] or Coussy [13]):

$$\text{div } \sigma = 0 \tag{1a}$$
$$\sigma = c : \epsilon(\vec{u}) - p\alpha \tag{1b}$$
$$\kappa = \alpha : \epsilon(\vec{u}) + \beta p \tag{1c}$$
$$\text{div } \vec{q} + \dot{\kappa} = 0 \tag{1d}$$
$$\vec{q} = -k\,\overrightarrow{\text{grad}}\,p \tag{1e}$$

where η denotes the porosity of the porous matrix. $\vec{u}$ is the displacement field and $\dot{\vec{u}}$ its time derivative, σ is the total Cauchy stress tensor and p is the pore pressure. $\vec{q} = \eta\left(\vec{v} - \dot{\vec{u}}\right)$ is the relative fluid flow, $\vec{v}$ being the velocity of the fluid. c is the fourth order tensor of elastic stiffness, α is the second order tensor of Biot coefficients, β is the Biot modulus and k is the permeability of the medium. κ denotes the variation - due to the displacement $\vec{u}$ - of the Lagrangean porosity, see Coussy [13], which reads in terms of the porosity of the porous matrix η and of its variation $\delta\eta$ due to the deformation of the medium:

$$\kappa = \delta\eta + \eta \text{ div } \vec{u}$$

Equations on the cracks. The cracks separating the porous parts of the medium are very soft and highly permeable. That means that the lips of the cracks can slide and open and, in order to maintain coherence, that the stress vector $\vec{T} = \sigma \,.\, \vec{n}$ is continuous on the cracks. The displacement field $\vec{u}$ is then discontinuous on the cracks and its jump $\vec{u}^+ - \vec{u}^-$ through a crack where $\vec{u}^+$ is the value of $\vec{u}$ on the side toward which $\vec{n}$ points and $\vec{u}^-$ is the value of $\vec{u}$ on the opposite side, is denoted by $[[\vec{u}]]$. The assumption of high permeability means that fluid pressure p is continuous on the cracks but the fluid flow is discontinuous, the jump of the normal flow is $[[\vec{q}]] \,.\, \vec{n}$ where $[[\vec{q}]]$ denotes the jump of $\vec{q}$ across the cracks.

According to these assumptions (see AppendixA), the poroelastic behavior of the cracks is modeled by the following equations:

$$\vec{T} = C \,.\, [[\vec{u}]] - p\vec{A} \tag{2a}$$
$$\kappa^c = \vec{A} \,.\, [[\vec{u}]] + Bp \tag{2b}$$
$$\frac{dQ}{ds} + [[\vec{q}]] \,.\, \vec{n} + \dot{\kappa}^c = 0 \tag{2c}$$
$$Q = -K\frac{dp}{ds} \tag{2d}$$

where s is the curvilinear abscissa along the cracks.

The set of equations (2) presents a structure similar to that of equations (1). (2a) is the elastic constitutive equation analogous to (1b), (1c) where κ^c is the variation of the Lagrangean porosity of the crack due to its sliding/opening.

The fluid pressure p is continuous through the cracks so the derivative $\frac{dp}{ds}$ can be written as $\frac{dp}{ds} = \overrightarrow{\text{grad}}\, p \,\boldsymbol{.}\, \vec{\tau}$ and the equation (2d) reads:

$$Q = -K\, \overrightarrow{\text{grad}}\, p \,\boldsymbol{.}\, \vec{\tau} \tag{3}$$

The sets (1) and (2) of equations need to be completed by boundary conditions and fluid mass balance equations at the crack junction points. As the purpose of this work is the bulk homogenization of the cracked poroelastic medium, the boundary conditions on $\partial\Omega$ are not relevant and are therefore not defined. Concerning the fluid mass balance equation, it is assumed that there is no point fluid source or well at the junction points of cracks, therefore the balance of fluid mass at junction points merely comes down to the (algebraic) sum of the flows coming from the cracks to the junction points, thus tending to zero.

2.1. Weak formulations

As stressed in the previous section, the topic of this work is the bulk homogenization of the cracked poroelastic medium and it is not necessary to precisely define the boundary conditions on $\partial\Omega$ (the boundary of Ω). So, the test fields considered in this section are taken as identically zero on $\partial\Omega$.

2.1.1. Mechanics

The weak formulation (virtual power formulation) of the mechanical equilibrium is obtained in the usual way: first by the scalar multiplication of the balance equation (1a) by a test field $\vec{w}$ (a virtual velocity field), second by integrating the product over an intact part of the porous medium Ω bounded either by a part of the boundary $\partial\Omega$ or cracks and third by modifying the integral $\int div\, \sigma.\vec{w}\, ds$ by an integration-by-parts and finally by summing all the obtained equations to get:

$$\forall \vec{w}, \vec{w} = 0 \text{ on } \partial\Omega\,,\; -\int_\Omega \sigma : \epsilon\left(\vec{w}\right)\, \mathrm{d}s - \int_\Gamma \vec{T} \,\boldsymbol{.}\, [[\vec{w}]]\, \mathrm{d}l = 0 \tag{4}$$

It has to be stressed that, contrary to what is usually done but consistently with the discontinuity of the displacement field $\vec{u}$ through the cracks, the integral $\int_\Gamma \vec{T} \,\boldsymbol{.}\, [[\vec{w}]]\, \mathrm{d}l$ over the cracks Γ is not eliminated by assuming that the velocity field $\vec{w}$ is continuous through those cracks.

2.1.2. Balance of fluid volume

The fluid volume balance is the same as the fluid mass balance since the fluid is considered as incompressible and it regards the balance of the fluid volume in the porous part (1d) and in the cracks (2c). The weak formulation of the balance of fluid volume in the porous part is obtained in the same way as the virtual power formulation of the equilibrium (4), i.e.:

$$\forall r, r = 0 \text{ on } \partial\Omega\,,\; -\int_\Omega \vec{q} \,\boldsymbol{.}\, \overrightarrow{\text{grad}} r\, \mathrm{d}s + \int_\Omega \dot{\kappa} r\, \mathrm{d}s - \int_\Gamma [[\vec{q}]] \,\boldsymbol{.}\, \vec{n} r\, \mathrm{d}l = 0 \tag{5}$$

where r represents the pressure test field. Differently from (4) where $\vec{T} = \sigma \,\boldsymbol{.}\, \vec{n}$ is continuous and $\vec{w}$ discontinuous through the cracks, in (5) $\vec{q}$ is discontinuous but r is continuous.
The same procedure along with the balance of flows at the junction points of the cracks yields to:

$$\forall r, r = 0 \text{ on } \partial\Omega\,,\; -\int_\Gamma Q \frac{dr}{ds} \mathrm{d}l + \int_\Gamma [[\vec{q}]]\, .\vec{n} r\, \mathrm{d}l + \int_\Gamma \dot{\kappa}^c r\, \mathrm{d}l = 0 \tag{6}$$

By adding the two previous weak formulations, we obtain:

$$\begin{aligned}\forall r, r = 0 \text{ on } \partial\Omega\,,\; &-\int_\Omega \vec{q} \,\boldsymbol{.}\, \overrightarrow{\text{grad}} r\, \mathrm{d}s - \int_\Gamma Q \frac{dr}{ds}\, \mathrm{d}l + \int_\Omega \dot{\kappa} r\, ds \\ &+ \int_\Gamma \dot{\kappa}^c r\, \mathrm{d}l + \int_{\partial\Omega} Q^b r\, \mathrm{d}l + \sum_i Q^i r\left(\vec{x}^i\right) = 0\end{aligned} \tag{7}$$

which, writing $\frac{dr}{ds} = \overrightarrow{gradr} \cdot \vec{\tau}$ reads:

$$\begin{aligned} \forall r, r = 0 \text{ on } \partial\Omega\,,\; - \int_{\Omega} \vec{q} \cdot \overrightarrow{\mathrm{grad}} r \,\mathrm{d}s - \int_{\Gamma} Q\vec{\tau} \cdot \overrightarrow{\mathrm{grad}} r \,\mathrm{d}l + \int_{\Omega} \dot{\kappa} r \,\mathrm{d}s \\ + \int_{\Gamma} \dot{\kappa}^c r \,\mathrm{d}l + \int_{\partial\Omega} Q^b r \,\mathrm{d}l + \sum_i Q^i r\left(\vec{x}^i\right) = 0 \end{aligned} \tag{8}$$

3. Homogenization

As presented in section 2, the considered method of up-scaling is based on asymptotic expansions, the small parameter e of those expansions being the ratio of the homothety mapping the cell Y onto the periods of the medium. This means that it is not only one medium that is considered but a sequence of media parametrized by e. Consequently, all the involved fields σ, $\vec{u}$, κ, $\vec{q}$, p, T, Q and κ^c depend on e. To underline this dependence, they are denoted by $\sigma^{(e)}$, $\vec{u}^{(e)}$, $\kappa^{(e)}$, $\vec{q}^{(e)}$, $p^{(e)}$, $T^{(e)}$, $Q^{(e)}$ and $\kappa^{c(e)}$.

The behavior of all the mechanical and porous characteristics of the porous matrix with respect to the small parameter e have to be detailed before implementing the asymptotic process since the homogenized modeling depends on this behavior. The mechanical and porous characteristics of the porous matrix, that is to say c, α, β and k are assumed to be locally periodic, i.e. their dependence of the space variable $\vec{x}$ takes the form $f\left(\vec{x}, \frac{\vec{x}}{e}\right)$ where the function $\vec{y} \in Y \longrightarrow f\left(\vec{x}, \vec{y}\right)$ is Y-periodic. On top of the local periodicity of the coefficients of the cracks and according to a similar study presented in Caillerie [9] for heat conduction, it is consistent to assume for them certain behaviours with respect to the small parameter e. As stressed out at the beginning of this section, the use of expansions implies that a sequence of media with thinner and thinner periods is considered. Consequently, these media present more and more cracks and if those cracks are not assumed to be stiffer and stiffer then the whole strain of the media will localise on the cracks. Conversely, if the permeability of the cracks is not assumed to go to zero with e then the whole flow will go through the cracks. In a similar way of thinking, the Biot's modulus of the cracks goes to zero like e in order that the macrocopic variation of the porosity should not be concentrated on the cracks. In the asymptotic process, it is then assumed that the modelling (2) of the cracks takes the form:

$$\vec{T}^{(e)} = \frac{C}{e} \cdot \left[\left[\vec{u}^{(e)}\right]\right] - p^{(e)}\vec{A} \tag{9a}$$

$$\kappa^{c(e)} = \vec{A} \cdot \left[\left[\vec{u}^{(e)}\right]\right] + eBp^{(e)} \tag{9b}$$

$$\frac{dQ^{(e)}}{ds} + \left[\left[\vec{q}^{(e)}\right]\right] .\vec{n} + \dot{\kappa}^{c(e)} = 0 \tag{9c}$$

$$Q^{(e)} = -eK\,\overrightarrow{\mathrm{grad}} p^{(e)} \cdot \vec{\tau} \tag{9d}$$

3.1. Asymptotic expansions

Let Γ^Y denote the set of all cracks of the cell Y.

We look for the solution $\vec{u}^{(e)}, p^{(e)}$ of the consolidation problem of the form of double scale asymptotic expansions:

$$\vec{u}^{(e)} = \vec{u}^{(0)}\left(\vec{x}\right) + e\vec{u}^{(1)}\left(\vec{x}, \frac{\vec{x}}{e}\right) + e^2\vec{u}^{(2)}\left(\vec{x}, \frac{\vec{x}}{e}\right) + \cdots \tag{10a}$$

$$p^{(e)} = p^{(0)}\left(\vec{x}\right) + ep^{(1)}\left(\vec{x}, \frac{\vec{x}}{e}\right) + e^2p^{(2)}\left(\vec{x}, \frac{\vec{x}}{e}\right) + \cdots \tag{10b}$$

where $\vec{u}^{(k)}\left(\vec{x}, \vec{y}\right)$ and $p^{(k)}\left(\vec{x}, \vec{y}\right)$ $k = 1, \dots$ are functions of the large scale variable $\vec{x} \in \Omega$ and of the small scale variable $\vec{y} \in Y$ and are Y-periodic with respect to $\vec{y}$. It can be noted that the first terms $\vec{u}^{(0)}$ and $p^{(0)}$ of those expansions are assumed not to depend on the variable $\vec{y}$. This is consistent with the general idea of the homogenization process which is to smooth out the fine-scale heterogeneities.

According to the expansions (10), those of the gradient of $\vec{u}^{(e)}$ of the strain $\epsilon\left(\vec{u}^{(e)}\right)$, of the gradient of the fluid pressure $p^{(e)}$ and of the jump $\left[\left[\vec{u}^{(e)}\right]\right]$ of $\vec{u}^{(e)}$ on cracks read:

$$\nabla\vec{u}^{(e)} = \nabla^x\vec{u}^{(0)} + \nabla^y\vec{u}^{(1)} + e\left(\nabla^x\vec{u}^{(1)} + \nabla^y\vec{u}^{(2)}\right) + \cdots \tag{11a}$$

$$\epsilon\left(\vec{u}^{(e)}\right) = \epsilon^x\left(\vec{u}^{(0)}\right) + \epsilon^y\left(\vec{u}^{(1)}\right) + e\left(\epsilon^x\left(\vec{u}^{(1)}\right) + \epsilon^y\left(\vec{u}^{(2)}\right)\right) + \cdots \tag{11b}$$

$$\overrightarrow{\mathrm{grad}}p^{(e)} = \overrightarrow{\mathrm{grad}}^x p^{(0)} + \overrightarrow{\mathrm{grad}}^y p^{(1)} + e\left(\overrightarrow{\mathrm{grad}}^x p^{(1)} + \overrightarrow{\mathrm{grad}}^y p^{(2)}\right) + \cdots \tag{11c}$$

$$\left[\left[\vec{u}^{(e)}\right]\right] = e\left[\left[\vec{u}^{(1)}\right]\right]\left(\vec{x},\frac{\vec{x}}{e}\right) + e^2\left[\left[\vec{u}^{(2)}\right]\right]\left(\vec{x},\frac{\vec{x}}{e}\right) + \cdots \tag{11d}$$

where the equations (11a), (11b) and (11c) hold true for $\vec{x}$ in the porous parts of Ω and the equation (11d) for $\vec{x}$ on the cracks Γ.
As already remarked, $\vec{u}^{(0)}$ of the expansion (10a) is assumed to be a smooth macroscopic displacement field. At this scale the cracks are smoothed out, consequently the expansion of the jump $\left[\left[\vec{u}^{(e)}\right]\right]$ begins at order 1 and, moreover, in the jumps $\left[\left[\vec{u}^{(k)}\right]\right]$, $k = 1, \ldots$ only the small scale variable $\vec{y}$ is concerned.

The constitutive equation (1b), the equation governing the variation of porosity (1c), the Darcy's law (1e) and the equations (9) along the cracks entails that the expansions of the stress $\sigma^{(e)}$, the variation of porosity $\delta\eta^{(e)}$ and the fluid flow $\vec{q}^{(e)}$ have to be of the following forms:

$$\sigma^{(e)}(\vec{x}) = \sigma^{(0)}\left(\vec{x},\frac{\vec{x}}{e}\right) + e\sigma^{(1)}\left(\vec{x},\frac{\vec{x}}{e}\right) + e^2\sigma^{(2)}\left(\vec{x},\frac{\vec{x}}{e}\right) + \cdots \tag{12a}$$

$$\kappa^{(e)}(\vec{x}) = \kappa^{(0)}\left(\vec{x},\frac{\vec{x}}{e}\right) + e\kappa^{(1)}\left(\vec{x},\frac{\vec{x}}{e}\right) + e^2\kappa^{(2)}\left(\vec{x},\frac{\vec{x}}{e}\right) + \cdots \tag{12b}$$

$$\vec{q}^{(e)}(\vec{x}) = \vec{q}^{(0)}\left(\vec{x},\frac{\vec{x}}{e}\right) + e\vec{q}^{(1)}\left(\vec{x},\frac{\vec{x}}{e}\right) + e^2\vec{q}^{(2)}\left(\vec{x},\frac{\vec{x}}{e}\right) + \cdots \tag{12c}$$

$$\vec{T}^{(e)}(\vec{x}) = \vec{T}^{(0)}\left(\vec{x},\frac{\vec{x}}{e}\right) + e\vec{T}^{(1)}\left(\vec{x},\frac{\vec{x}}{e}\right) + \cdots \tag{12d}$$

$$\kappa^{c(e)}(\vec{x}) = e\kappa^{c(1)}\left(\vec{x},\frac{\vec{x}}{e}\right) + e^2\kappa^{c(2)}\left(\vec{x},\frac{\vec{x}}{e}\right) + \cdots \tag{12e}$$

$$Q^{(e)}(\vec{x}) = eQ^{(1)}\left(\vec{x},\frac{\vec{x}}{e}\right) + e^2Q^{(2)}\left(\vec{x},\frac{\vec{x}}{e}\right) + \cdots \tag{12f}$$

Equations (12a), (12b) and (12c) hold true for $\vec{x}$ in the porous parts of Ω and equations (12d), (12e) and (12f) for $\vec{x}$ on the cracks Γ.

As a consequence of (12c), the jump of the fluid flow through the cracks expands into:

$$\left[\left[\vec{q}^{(e)}\right]\right](\vec{x}) = \left[\left[\vec{q}^{(0)}\right]\right]\left(\vec{x},\frac{\vec{x}^c}{e}\right) + e\left[\left[\vec{q}^{(1)}\right]\right]\left(\vec{x},\frac{\vec{x}}{e}\right) + e^2\left[\left[\vec{q}^{(2)}\right]\right]\left(\vec{x},\frac{\vec{x}}{e}\right) + \cdots \tag{13}$$

The choices of the order of magnitude of the characteristics of the cracks with respect to e in (9) are such that in the expansions (12), the leading term of the stress vector $\vec{T}^{(e)}$ is of the order of e^0 and not of e and that those of $\kappa^{c(e)}$ and $Q^{(e)}$ are of the order of e and not of e^0.

3.2. Balance equations

3.2.1. Preliminary results

To obtain the macroscopic balance equations (of momentum and of fluid flow) we use a lemma established by Sanchez-Palencia in [25, (pp. 77-78)] that reads:

For a function $F(\vec{x},\vec{y})$ of $\vec{x}$ and $\vec{y}$, $Y-$periodic in $\vec{y}$, we have:

$$\lim_{e\to 0}\int_\Omega F\left(\vec{x},\frac{\vec{x}}{e}\right)ds_x = \int_\Omega \langle F\rangle(\vec{x})\,ds_x$$

where:

$$\langle F\rangle(\vec{x})=\frac{1}{|Y|}\int_Y F(\vec{x},\vec{y})\,\mathrm{d}s_y$$

This result is used in the following but it has to be extended to deal with integrals of the form $\int_\Gamma F\left(\vec{x},\frac{\vec{x}}{e}\right)\mathrm{d}l_x$ defined on Γ, i.e.:

For a function $F(\vec{x},\vec{y})$ of $\vec{x}$ and $\vec{y}$, $Y-$periodic in $\vec{y}$, we have:

$$\lim_{e\to 0}\int_\Gamma F\left(\vec{x},\frac{\vec{x}}{e}\right)dl_x=\int_\Omega\langle F\rangle(\vec{x})\,ds_x$$

where:

$$\langle F\rangle(\vec{x})=\frac{1}{|Y|}\int_{\Gamma^Y} F(\vec{x},\vec{y})\,\mathrm{d}l_y$$

(Γ^y denotes the set of cracks of the cell Y).

Using the idea and the notations presented in Sánchez-Palencia [25], the integral $\int_\Gamma F\left(\vec{x},\frac{\vec{x}}{e}\right)dl_x$ is split into:

$$\int_\Gamma F\left(\vec{x},\frac{\vec{x}}{e}\right)dl_x=\sum_{\text{periods}}\int_{e\Gamma^Y}F\left(\vec{x},\frac{\vec{x}}{e}\right)dl_x$$

The change of variables $\vec{x}\leftrightarrow e\vec{y}$ in the integral $\int_{e\Gamma_Y}F\left(\vec{x},\frac{\vec{x}}{e}\right)dl_x$ yields (note that $dl_x=edl_y$):

$$\int_\Gamma F\left(\vec{x},\frac{\vec{x}}{e}\right)dl_x=\sum_{\text{periods}}e\int_{\Gamma^Y}F(\vec{x},\vec{y})\,dl_y$$

so:

$$e\int_\Gamma F\left(\vec{x},\frac{\vec{x}}{e}\right)dl_x=\sum_{\text{periods}}e^2\,|Y|\,\langle F\rangle(\vec{x})$$

now that sum over the periods can be seen as a Riemann's sum of the integral $\int_\Omega\langle F\rangle(\vec{x})\,ds_x$ hence the result when $e\searrow 0$.

3.2.2. Mechanics

We consider in the formulation (4) a smooth (differentiable) macroscopic virtual velocity field $\vec{w}$ which vanishes on $\partial\Omega$. It is worth noting that the continuity of $\vec{w}$ in Ω means that the jumps $[[\vec{w}]]$ through the cracks are zero. So, using lemma 3.2.1, the limit $e\searrow 0$, yields:

$$\forall\vec{w},\vec{w}=0\text{ on }\partial\Omega\,,\ -\int_\Omega\frac{1}{|Y|}\left(\int_Y\sigma^{(0)}\,\mathrm{d}s_y\right):\epsilon(\vec{w})\,\mathrm{d}s_x=0$$

Defining the average stress tensor $\langle\sigma\rangle$ as:

$$\langle\sigma\rangle=\frac{1}{|Y|}\int_Y\sigma^{(0)}\,\mathrm{d}s_y\tag{14}$$

the previous virtual power formulation of the macroscopic equilibrium reads:

$$\forall\vec{w},\vec{w}=0\text{ on }\partial\Omega\,,\ -\int_\Omega\langle\sigma\rangle:\epsilon(\vec{w})\,\mathrm{d}s_x=0\tag{15}$$

which proves that:

$$\operatorname{div}\langle\sigma\rangle=0\text{ in }\Omega\tag{16}$$

3.2.3. Fluid flow

Taking r in (8) as a smooth macroscopic function which vanishes on $\partial\Omega$ and making $e \searrow 0$ with the use of lemma 3.2.1, we get:

$$\forall r, r = 0 \text{ on } \partial\Omega\,,\ -\int_\Omega \frac{1}{|Y|}\left(\int_Y \vec{q}^{(0)}\,\mathrm{d}s_y + \int_{\Gamma^Y} Q^{(1)}\vec{\tau}\,\mathrm{d}l_y\right)\centerdot \overrightarrow{grad}^x r\,\mathrm{d}s_x$$
$$+\int_\Omega \frac{1}{|Y|}\left(\int_Y \dot{\kappa}^{(0)}\mathrm{d}s_y + \int_{\Gamma^Y}\left(\dot{\kappa}^{c(1)}\right)\mathrm{d}l_y\right) r\,\mathrm{d}s_x = 0$$

Defining the average flow vector $\langle\vec{q}\rangle$ and the variation of lagrangean porosity $\langle\kappa\rangle$ as:

$$\langle\vec{q}\rangle = \frac{1}{|Y|}\left(\int_Y \vec{q}^{(0)}\,\mathrm{d}s_y + \int_{\Gamma^Y} Q^{(1)}\vec{\tau}\,\mathrm{d}l_y\right) \tag{17}$$

$$\langle\kappa\rangle = \frac{1}{|Y|}\left(\int_Y \kappa^{(0)}\,\mathrm{d}s_y + \int_{\Gamma^Y} \kappa^{c(1)}\,\mathrm{d}l_y\right) \tag{18}$$

we get:

$$\forall r, r = 0 \text{ on } \partial\Omega\,,\ -\int_\Omega \frac{1}{|Y|}\langle\vec{q}\rangle\centerdot\overrightarrow{grad}^x r\,\mathrm{d}s_x + \int_\Omega \langle\kappa\rangle\, r\,\mathrm{d}s_x = 0 \tag{19}$$

which entails that:

$$\mathrm{div}^x\,\langle\vec{q}\rangle + \langle\dot{\kappa}\rangle = 0 \tag{20}$$

3.3. Expansions of the constitutive equations

The expansions of the equations (1b), (1c), (1e), (9a), (9b) and (9d) yield at the lowest order:

$$\sigma^{(0)} = c : \left(\epsilon^x\left(\vec{u}^{(0)}\right) + \epsilon^y\left(\vec{u}^{(1)}\right)\right) - p^{(0)}\alpha \tag{21a}$$

$$\kappa^{(0)} = \alpha : \left(\epsilon^x\left(\vec{u}^{(0)}\right) + \epsilon^y\left(\vec{u}^{(1)}\right)\right) + \beta p^{(0)} \tag{21b}$$

$$\vec{q}^{(0)} = -k\left(\overrightarrow{\mathrm{grad}}^x p^{(0)} + \overrightarrow{\mathrm{grad}}^y p^{(1)}\right) \tag{21c}$$

$$\vec{T}^{(0)} = C\centerdot\left[\left[\vec{u}^{(1)}\right]\right] - p^{(0)}\vec{A} \tag{21d}$$

$$\kappa^{c(1)} = \vec{A}\centerdot\left[\left[\vec{u}^{(1)}\right]\right] + Bp^{(0)} \tag{21e}$$

$$Q^{(1)} = -K\left(\overrightarrow{\mathrm{grad}}^x p^{(0)} + \overrightarrow{\mathrm{grad}}^y p^{(1)}\right)\centerdot\vec{\tau} \tag{21f}$$

3.4. Self-balanced problems on the cell Y

The macroscopic balance equations (16) and (20) or alternatively (15) and (19) are not sufficient to define the macroscopic homogenized modeling of the cracked porous medium, macroscopic constitutive equations are needed. Those macroscopic constitutive equations come from the averaging over Y and/or Γ^Y of the equations (21). But those first terms of the expansions of the constitutive equations of the porous medium involve not only the macroscopic fields $\vec{u}^{(0)}$ and $p^{(0)}$ but also $\vec{u}^{(1)}$ and $p^{(1)}$ which depend on $\vec{y}$ and are unknown. Therefore, previous to the averaging of (21), $\vec{u}^{(1)}$ and $p^{(1)}$ must be determined in terms of the macroscopic fields $\vec{u}^{(0)}$ and $p^{(0)}$ (see Sánchez-Palencia [25] or Bensoussan et al. [6] for more details). The so called self-balanced problems, i.e. the problems enabling to determine $\vec{u}^{(1)}$ and $p^{(1)}$, are obtained by taking in the weak formulations (4) and (8) suitable test fields of the form $\vec{w} = \theta\left(\vec{x}\right)\vec{v}\left(\frac{\vec{x}}{e}\right)$ and $r = \theta\left(\vec{x}\right)t\left(\frac{\vec{x}}{e}\right)$. This choice, unlike the test fields considered in section 3.2, do not smear out the heterogeneities but emphasizes the dependence on the variable $\vec{y}$ (see Sánchez-Palencia [25, page 79] for a more complete presentation).

3.4.1. Mechanics

Let $\vec{w}$ be the field $\vec{w} = \theta\left(\vec{x}\right)\vec{v}\left(\frac{\vec{x}}{e}\right)$ where $\vec{v}\left(y\right)$ is a periodic function defined on Y and $\theta\left(\vec{x}\right)$ is a smooth macroscopic function which vanishes on $\partial\Omega$, we have:

$$\epsilon\left(\vec{w}\right) = \left(\vec{v}\otimes\overrightarrow{\mathrm{grad}}^x\theta\right)^S + \frac{1}{e}\theta\,\epsilon^y\left(\vec{v}\right)$$

Taking $\vec{w} = \theta\left(\vec{x}\right)\vec{v}\left(\frac{\vec{x}}{e}\right)$ in (4) and making $e\searrow 0$ with the use of lemmas 3.2.1 and 3.2.1, yield:

$$\forall\theta,\vec{v},\theta=0 \text{ on } \partial\Omega\,,\ -\int_\Omega\frac{1}{|Y|}\left(\int_Y\sigma^{(0)}:\epsilon^y\left(\vec{v}\right)\mathrm{d}s_y + \int_{\Gamma^Y}\vec{T}^{(0)}\centerdot[[\vec{v}]]\,\mathrm{d}l_y\right)\theta\,\mathrm{d}s_x = 0$$

as θ is any smooth macroscopic field, that entails:

$$\forall\vec{v},Y\text{-periodic}\,,\ \int_Y\sigma^{(0)}:\epsilon^y\left(\vec{v}\right)\mathrm{d}s_y + \int_{\Gamma^Y}\vec{T}^{(0)}\centerdot[[\vec{v}]]\,\mathrm{d}l_y = 0$$

which is the weak formulation of the mechanical self-balanced of the cell Y. Taking into account the constitutive equations (21a) and (21d), yields to the weak formulation of the mechanical self-balanced problem, i.e.:

$$\begin{aligned}&\text{Given } \epsilon^x\left(\vec{u}^{(0)}\right) \text{ and } p^{(0)}\text{, find } \vec{u}^{(1)}\left(\vec{x},\vec{y}\right),\ Y\text{-periodic, such that:}\\ &\forall\vec{v},Y\text{-periodic}\,,\ \int_Y\left(c:\left(\epsilon^x\left(\vec{u}^{(0)}\right)+\epsilon^y\left(\vec{u}^{(1)}\right)\right)-p^{(0)}\alpha\right):\epsilon^y\left(\vec{v}\right)\mathrm{d}s_y\\ &\qquad + \int_{\Gamma^Y}\left(C\centerdot\left[\left[\vec{u}^{(1)}\right]\right]-p^{(0)}\vec{A}\right)\centerdot[[\vec{v}]]\,\mathrm{d}l_y = 0\end{aligned} \tag{22}$$

The strong form of the problem reads:

$$\mathrm{div}^y\sigma^{(0)} = 0\,,\ \text{in } Y \tag{23a}$$
$$\sigma^{(0)}\centerdot\vec{n} = \vec{T}^{(0)}\,,\ \text{on } \Gamma^Y \tag{23b}$$
$$\sigma^{(0)} = c:\left(\epsilon^x\left(\vec{u}^{(0)}\right)+\epsilon^y\left(\vec{u}^{(1)}\right)\right)-p^{(0)}\alpha \tag{23c}$$
$$\vec{T}^{(0)} = C\centerdot\left[\left[\vec{u}^{(1)}\right]\right]-p^{(0)}\vec{A},\ \text{on } \Gamma^Y \tag{23d}$$
$$\vec{u}^{(1)},\sigma^{(0)}\ Y\text{-periodic} \tag{23e}$$

It is worth noting that this problem is purely mechanical in the sense that the only unknowns are the displacement field $\vec{u}^{(1)}$ and the stress field $\sigma^{(0)}$, the pressure field $p^{(1)}$ and the flows $\vec{q}^{(0)}$ and $Q^{(1)}$ being not involved.

3.4.2. Fluid flow

Let r be the field $r = \theta\left(\vec{x}\right)w\left(\frac{\vec{x}}{e}\right)$ where $w\left(\vec{y}\right)$ is a periodic function defined on Y and $\theta\left(\vec{x}\right)$ is a smooth macroscopic function which vanishes on $\partial\Omega$. We then have:

$$\overrightarrow{\mathrm{grad}}\,r = w\,\overrightarrow{\mathrm{grad}}^x\theta + \frac{1}{e}\theta\,\overrightarrow{\mathrm{grad}}^y w$$

Taking $r = \theta\left(\vec{x}\right)w\left(\frac{\vec{x}}{e}\right)$ in (8) and making $e\searrow 0$, yield:

$$\forall\theta,w\,,\ -\int_\Omega\frac{1}{|Y|}\left(\int_Y\vec{q}^{(0)}\centerdot\overrightarrow{\mathrm{grad}}^y w\,\mathrm{d}s_y + \int_{\Gamma^Y}Q^{(1)}\overrightarrow{\mathrm{grad}}^y w\centerdot\vec{\tau}\,\mathrm{d}l_y\right)\theta\,\mathrm{d}s_x = 0$$

as θ is any smooth macroscopic field, that entails:

$$\forall w\,Y\text{-periodic}\,,\ \int_Y\vec{q}^{(0)}\centerdot\overrightarrow{\mathrm{grad}}^y w\,\mathrm{d}s_y + \int_{\Gamma^Y}Q^{(1)}\overrightarrow{\mathrm{grad}}^y w\centerdot\vec{\tau}\,\mathrm{d}l_y = 0$$

which is the weak form of the fluid volume self-balanced of the cell Y. Taking into account relations (21c) and (21f), yields to the weak formulation of the self-balanced filtration problem that reads:

Given $\overrightarrow{\text{grad}}^x p^{(0)}$, find $p^{(1)}(\vec{x}, \vec{y})$, Y-periodic, such that:

$$\begin{aligned}\forall w\, Y\text{-periodic}\,, \int_Y k \left(\overrightarrow{\text{grad}}^x p^{(0)} + \overrightarrow{\text{grad}}^y p^{(1)}\right) \cdot \overrightarrow{\text{grad}}^y w \, \mathrm{d}s_y \\ + \int_{\Gamma^Y} \left(K \left(\overrightarrow{\text{grad}}^x p^{(0)} + \overrightarrow{\text{grad}}^y p^{(1)}\right) \cdot \vec{\tau}\right) \left(\overrightarrow{\text{grad}}^y w \cdot \vec{\tau}\right) \mathrm{d}l_y = 0\end{aligned} \quad (24)$$

The strong form of the problem reads:

$$\begin{aligned}
\text{div}^y \, \vec{q}^{(0)} &= 0\,, \text{ in } Y \\
\frac{dQ^{(1)}}{ds} + \left[\left[\vec{q}^{(0)}\right]\right] \cdot \vec{n} &= 0\,, \text{ on } \Gamma^Y \\
\vec{q}^{(0)} &= -k \left(\overrightarrow{\text{grad}}^x p^{(0)} + \overrightarrow{\text{grad}}^y p^{(1)}\right) \\
Q^{(1)} &= -K \left(\overrightarrow{\text{grad}}^x p^{(0)} + \overrightarrow{\text{grad}}^y p^{(1)}\right) \cdot \vec{\tau} \\
\sum Q^{(1)} &= 0 \text{ at junction points of cracks} \\
p^{(1)}, \vec{q}^{(0)}, Q^{(1)} & \; Y\text{-periodic}
\end{aligned}$$

In a similar way to section 3.4.1, the problem (24) is a pure filtration one, the displacement field $\vec{u}^{(1)}$ and the stress field $\sigma^{(0)}$ are not involved.

So, at the microscopic scale, i.e. the cell Y-scale, the elasticity and the fluid flow problems are completely uncoupled.

3.5. Macroscopic constitutive equations

As problems (22) and (24) are linear, it is standard (see Sánchez-Palencia [25]) or (Bensoussan et al. [6]) to prove that $\vec{u}^{(1)}$ linearly depends on $e^x\left(\vec{u}^{(0)}\right)$ and $p^{(0)}$, and $p^{(1)}$ on $\overrightarrow{\text{grad}}^x p^{(0)}$. Consequently, by the constitutive equation (21b) and the expression (21b), $\sigma^{(0)}$ and $\kappa^{(0)}$ depend linearly on $e^x\left(\vec{u}^{(0)}\right)$ and $p^{(0)}$ and so does $\langle\sigma\rangle$ and $\langle\kappa\rangle$ defined by (14) and (18). In a similar way, it can be seen that $\langle\vec{q}\rangle$ depends linearly on $\overrightarrow{\text{grad}}^x p^{(0)}$. Therefore, the macroscopic constitutive equations read:

$$\langle\sigma\rangle = c^H : \epsilon^x\left(\vec{u}^{(0)}\right) - p^{(0)}\alpha^H \quad (25a)$$

$$\langle\kappa\rangle = \tilde{\alpha}^H : \epsilon^x\left(\vec{u}^{(0)}\right) + \beta^H p^{(0)} \quad (25b)$$

$$\langle\vec{q}\rangle = -k^H \overrightarrow{\text{grad}}^x p^{(0)} \quad (25c)$$

From those equations and (14), it is obvious that c^H_{ijkh} is the average of $\sigma^{(0)}_{ij}$ for $p^{(0)} = 0$ and $\epsilon^x\left(\vec{u}^{(0)}\right) = \left(\vec{e}_k \otimes \vec{e}_h\right)^S$ and that α^H_{ijk} is the average of $-\sigma^{(0)}_{ij}$ for $p^{(0)} = 1$ and $\epsilon^x\left(\vec{u}^{(0)}\right) = 0$. In the same way, k^H_{ij} is equal to $\langle\vec{q}\rangle_i$ (see (17)) for $\overrightarrow{\text{grad}}^x p^{(0)} = \vec{e}_j$.

On the standard energy-based assumptions of symmetries for the elastic tensors c and C namely:

$$c_{ijkl} = c_{klij} \text{ and } C_{ij} = C_{ji}$$

it is classical (see Sánchez-Palencia [25] or Bensoussan et al. [6]) to prove that c^H, k^H, α^H and $\tilde{\alpha}^H$ satisfy the following relations:

$$\begin{aligned}
c^H_{ijkl} &= c^H_{klij} \\
k^H_{ij} &= k^H_{ji} \\
\tilde{\alpha}^H &= \alpha^H
\end{aligned}$$

4. Damage

In this section, cracks are considered damageable, the purpose being to build the corresponding macroscopic modeling of the cracked porous medium.

4.1. Damage parameter and evolution law

$d^{(e)}$ being the damage parameter of the cracks, the constitutive equations of those cracks (9a) and (9d) are changed to:

$$\vec{T}^{(e)} = \left(1 - d^{(e)}\right) \frac{C}{e} \centerdot \left[\left[\vec{u}^{(e)}\right]\right] - p^{(e)} \vec{A} \tag{26}$$

$$Q^{(e)} = -eK\left(d^{(e)}\right) \overrightarrow{\text{grad}}\, p^{(e)} \centerdot \vec{\tau} \tag{27}$$

It can be stressed that, in the considered damage modeling, the damage parameter modifies only the stiffness C and not $\vec{A}$. This can be seen as purely heuristical but this choice seems consistent with the double scale asymptotic analysis of an elastic saturated porous matrix leading to Biot's modeling, see Auriault and Sanchez-Palencia [2] and Auriault [4]. On the other hand, the dependence of the permeability K with respect to $d^{(e)}$ is completely phenomenological (e.g.Rastiello et al. [24]).

The damage of the cracks is due to the opening and the shearing of the cracks. For the sake of simplicity, any difference between the opening and the closing of the cracks are disregarded and the evolution of the damage parameter d is given by:

$$d^{(e)}(t) = \sup_{0 \le \tau \le t} f\left(\frac{\| \left[\left[\vec{u}^{(e)}\right]\right](\tau) \|}{D^{(e)}}\right) \tag{28}$$

where f is the function:

$$z \xrightarrow{f} f(z) = \begin{cases} z(2-z) & 0 \le z < 1 \\ 1 & 1 \le z \end{cases} \tag{29}$$

and $D^{(e)}$ is a length-like feature of the material of the cracks. At initial time $t = 0$, the porous medium is assumed to be unloaded, unstressed, unstrained and undamaged which means that the initial value of the damage parameter is 0.

$d^{(e)}(t)$ is a function of the history $\left\{\left[\left[\vec{u}^{(e)}\right]\right](\tau)\ ;\ \tau \le t\right\}$ of $\left[\left[\vec{u}^{(e)}\right]\right]$ up to time t. Its evolution is governed only by the opening or shearing of the cracks and not directly by the fluid pressure.

4.2. Homogenization of the damageable cracked porous medium

The damage parameter is sought in the form of the double scale asymptotic expansion:

$$d^{(e)}(\vec{x}, t) = d^{(0)}\left(\vec{x}, \frac{\vec{x}}{e}, t\right) + ed^{(1)}\left(\vec{x}, \frac{\vec{x}}{e}, t\right) + e^2 d^{(2)}\left(\vec{x}, \frac{\vec{x}}{e}, t\right) + \cdots$$

Almost all the analysis and equations of section 3 remain valid. Equation (23d) is slightly modified to:

$$\vec{T}^{(0)} = \left(1 - d^{(0)}\right) C \centerdot \left[\left[\vec{u}^{(1)}\right]\right] - p^{(0)} \vec{A}$$

but all the other equations of (23) remain unmodified. The main change lies in the nature of the mechanical self-balanced problem (23) on the cell Y which, in section 3.4 is a purely time independent elastic problem. In this section, due to damage, it becomes a non-linear quasi-static evolution, involving an evolution law for the damage parameter $d^{(0)}$ which comes from the expansion (28).

As, according to (11d), the first term of the expansion of $\left[\left[\vec{u}^{(e)}\right]\right]$ is of the order of e, it is consistent to assume that $D^{(e)}$ is proportional to e:

$$D^{(e)} = eD$$

With this assumption, it can be proved (see AppendixB) that the evolution law for $d^{(0)}$ reads:

$$d^{(0)}(t) = \sup_{0\leq\tau\leq t} f\left(\frac{\|\,[[\vec{u}^{(1)}]]\,(\tau)\,\|}{D}\right) \tag{30}$$

In the case of damageable cracks, the weak formulation of the mechanical self-balanced problem reads:

$$\begin{aligned}
&\text{Given the histories } \left\{\epsilon^x\left(\vec{u}^{(0)}\right)(\tau)\ ;\ 0\leq\tau\leq t\right\} \text{ and } \left\{p^{(0)}(\tau)\ ;\ 0\leq\tau\leq t\right\},\\
&\qquad\text{find } \vec{u}^{(1)}(\vec{x},\vec{y},\tau) \text{ and } d^{(0)}(\vec{x},\vec{y},\tau)\,,\ Y\text{-periodic}, \tau\in[0,t]\,, \text{ such that:}\\
&\forall\tau\in[0,t]\,,\forall\vec{v}, Y\text{-periodic}\,,\ \int_Y\left(c:\left(\epsilon^x\left(\vec{u}^{(0)}(\tau)\right)+\epsilon^y\left(\vec{u}^{(1)}(\tau)\right)\right)-p^{(0)}(\tau)\,\alpha\right):\epsilon^y(\vec{v})\,\mathrm{d}s_y\\
&\qquad+\int_{\Gamma^Y}\left(\left(1-d^{(0)}(\tau)\right)C\cdot\left[\left[\vec{u}^{(1)}(\tau)\right]\right]-p^{(0)}(\tau)\,\vec{A}\right)\cdot[[\vec{v}]]\,\mathrm{d}l_y=0\\
&\qquad\qquad\text{with } d^{(0)}(\tau)=\sup_{0\leq\rho\leq\tau} f\left(\frac{\|\,[[\vec{u}^{(1)}(\rho)]]\,\|}{D}\right)
\end{aligned} \tag{31}$$

The solution of this quasi-static evolution problem, allows the stress $\sigma^{(0)}(t)$ and by (14) the macroscopic stress $\langle\sigma\rangle(t)$ to be determined. Thus, the macroscopic constitutive equation reads:

$$\left(\left\{\epsilon^x\left(\vec{u}^{(0)}\right)(\tau)\ ;\ 0\leq\tau\leq t\right\},\left\{p^{(0)}(\tau)\ ;\ 0\leq\tau\leq t\right\}\right)\longrightarrow\langle\sigma\rangle(t)$$

The strong form of problem (31) can be obtained in the same way as (23) is obtained from (22).

The solution of the evolution problem (31) yields to $d^{(0)}(t)$, as a functional of the histories $\left\{\epsilon^x\left(\vec{u}^{(0)}\right)(\tau)\ ;\ 0\leq\tau\leq t\right\}$ and $\left\{p^{(0)}(\tau)\ ;\ 0\leq\tau\leq t\right\}$. The self balanced filtration problem (24) remains the same with $K\left(d^{(0)}\right)$ instead of K.

5. Numerical implementation of the linear case

The identification and the use of the macroscopic constitutive equations of the cracked porous medium require the solution of the (weak formulation) problems (22) for the mechanical part - or (31) in the case of damageable cracks - and (24) for the flow part. However, even for simple configurations of the elementary cell, those problems cannot be solved analytically and a numerical FEM computation is needed. For the sake of simplicity, the numerical computation is carried out in 2D.

5.1. Cell and material properties

In order to carry out the numerical computation, an elementary cell (or REV) has to be defined. The considered cell is depicted in fig.2; the same cell as Marinelli [19] has been adopted, the dimensions are the unity for both sides. The grey parts represent the poroelastic, homogeneous isotropic matrix (Lamï¿œ constants: $\lambda = 1,442\cdot10^9$ Pa and $\mu = 0,961\cdot10^9$ Pa , i.e. Young's modulus $E = 2,5\cdot10^9$ Pa and Poisson's ratio $\nu = 0,3$), corresponding to a typical clay rock material. The Biot's tensor of the poroelastic matrix is proportional to the identity: $\alpha = 0,4\,\mathbb{I}$; its transmisivity K is assumed to be 10^{-7}m^2/Pa i.e. this is equivalent to assume a permeability k^h equal to 10^{-10}m/Pa with a crack thickness 0,001 times the cell (unit) length.[1]

Assuming that the material of the cracks is isotropic and that its Biot's tensor is proportional to the identity (see (A.6) and (A.7)), the stiffness tensor of the crack and its Biot's vector respectively read $C = C_T\vec{\tau}\otimes\vec{\tau} + C_N\vec{n}\otimes\vec{n}$ and $\vec{A} = A\vec{n}$. In the following, a parametric study on the stiffnesses C_T and C_N, the Biot's coefficient A and the permeability of the crack k^h is proposed in order to study the influence of the characteristics of the cracks on the equivalent macroscopic poroelastic medium (note that $C_T = C_N = C$).

[1]More details are given in appendix A. Very thin layer of a deformable porous medium.

Remark 1. Due to the symmetries of the elementary cell with respect to the two axes and the homogeneity and isotropy of the poroelastic medium and cracks, some properties of the homogenized medium are expected to be isotropic. So, following Léné and Duvaut [18] or Caillerie [10] or AppendixC for the quasi static non linear case of damageable cracks, the homogenized medium is orthotropic, the stiffnesses c^H_{1112}, c^H_{2212} (and all those ensued by the usual symmetries of indices) are zero, and the the Biot's matrix α^H is diagonal ($\alpha^H_{12} = \alpha^H_{21} = 0$). It has to be noted that, in general, the homogenized material is just orthotropic and not isotropic. In the same way, the homogenized permeability matrix is diagonal.

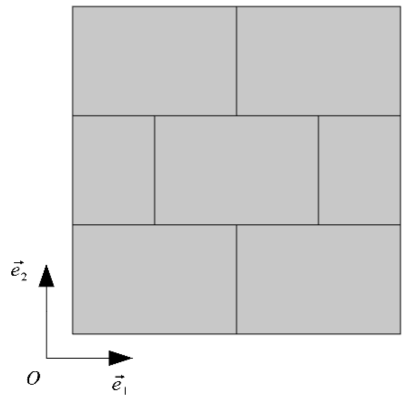

Figure 2: Elementary cell geometry, see [19].

In this following section, the self equilibrium problem of the cell (22) is numerically solved. The solutions of a 3+1 boundary value problem (three for the elasticity problem and one for the water pressure), are integrated and the homogenized coefficients are obtained in the macroscale [2].

5.2. Numerical validation: mechanics

5.2.1. Macroscopic elastic stifnesses

According to section 3.5, the macroscopic stiffnesses are obtained through the solution of problem (22) for $p^{(0)} = 0$ and for $\epsilon^x\left(u^{(0)}\right)$, i.e. pure elongations in the direction 1 and 2 and a simple shear, thus imposing a strain matrix as follows :

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} ; \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} ; \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

For those computations, the crack stiffnesses are $C_T = C_N = C = 10^{12}$Pa/m.

Results of the displacement and Von Mises stress field at the micro-scale are given in fig. 3. As expected, despite the isotropic elasticity at the micro-level, the homogenized solution is not isotropic due to the cell geometry.

This result is quantitatively consistent with [19].

A parametric study on the influence of the cracks stiffness C in the range from 10^6Pa/m to 10^{15}Pa/m is here proposed. The elastic characteristics of the porous matrix remain unchanged.

Results are presented in fig.4.

Consistently with the remark 1 about the orthotropy of the homogenized medium, the coefficients c^H_{1112}, c^H_{1211}, c^H_{1222} and c^H_{2212} [3] are found numerically null. For a small enough crack stiffness C, the dependence of

[2]The following Cauchy stress and Biot problem notation is considered:

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{pmatrix} = \begin{pmatrix} C_{1111} & C_{1122} & C_{1112} \\ C_{1122} & C_{2222} & C_{2212} \\ C_{1112} & C_{2212} & C_{1212} \end{pmatrix} \begin{pmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{12} \end{pmatrix} \text{ and } \overrightarrow{A}^H = \begin{pmatrix} A_{1111} \\ A_{2222} \\ A_{1212} \end{pmatrix}$$

[3]Notation in the Cauchy relation:

$$\begin{pmatrix} \langle\sigma_{11}\rangle \\ \langle\sigma_{22}\rangle \\ \langle\sigma_{12}\rangle \end{pmatrix} = \begin{pmatrix} c^H_{1111} & c^H_{1122} & c^H_{1112} \\ c^H_{2211} & c^H_{2222} & c^H_{2212} \\ c^H_{1211} & c^H_{1222} & c^H_{1212} \end{pmatrix} \cdot \begin{pmatrix} \epsilon^x_{11} \\ \epsilon^x_{22} \\ \epsilon^x_{12} \end{pmatrix} \tag{32}$$

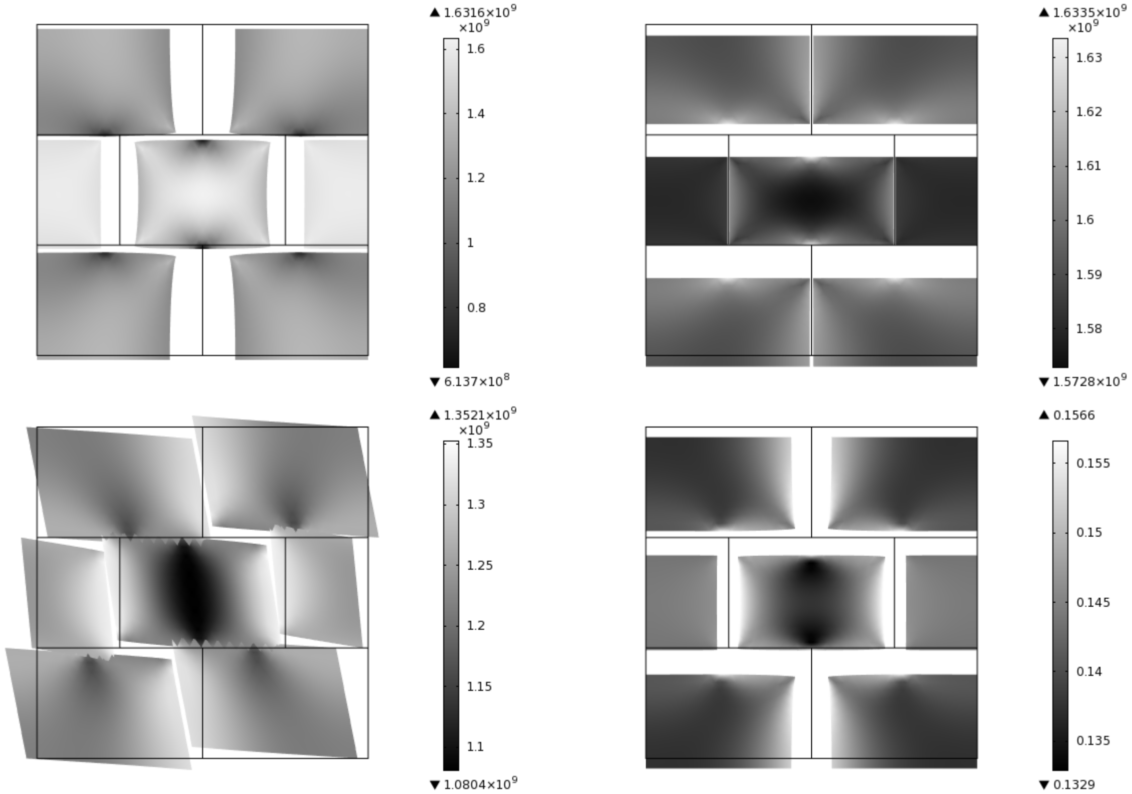

Figure 3: Von Mises stress fields N/mm^2 in the cell for loadings in the 4 degrees of freedom: $lm = 11, 22, 12$ and $p^{(0)}$.

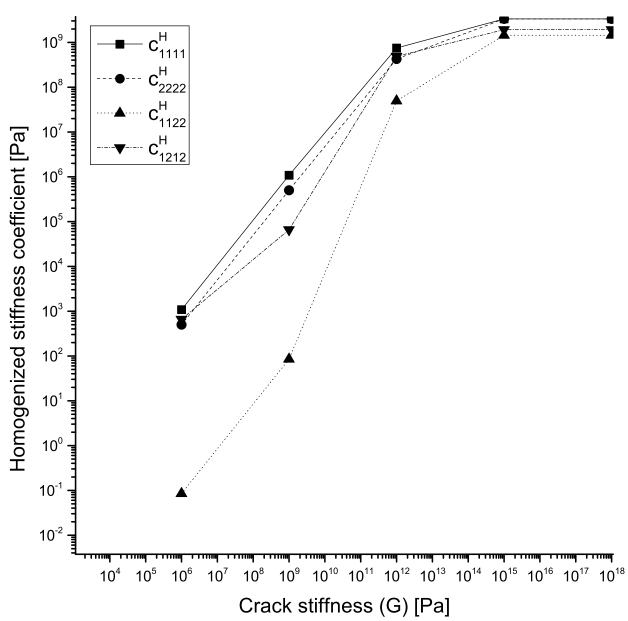

Figure 4: Homogenized coefficients vs. crack stiffness evolution

the homogenized coefficients on C is almost linear, i.e. the overall stiffness is essentially controlled by that of the cracks and is independent from the porous material properties. It is obvious that the cracked porous medium is weaker than the porous medium itself since when a stress is applied, the overall strain is larger than what it would be in the intact porous medium. It is obvious too that the overall stiffness is close to that of the porous medium if the cracks are very stiff; indeed, in that case, the porous medium is almost continuous. That is numerically verified, for a crack stiffness G higher than $1 \cdot 10^{15} Pa$, the homogenized medium is isotropic, its Young's modulus and Poisson's ratio being those of the porous medium.

$$\underline{\underline{c}}^H = \begin{pmatrix} 3,36 \cdot 10^9 & 1,44 \cdot 10^9 & 0 \\ 1,44 \cdot 10^9 & 3,36 \cdot 10^9 & 0 \\ 0 & 0 & 1,92 \cdot 10^9 \end{pmatrix} = \begin{pmatrix} \lambda + 2\mu & \lambda & 0 \\ \lambda & \lambda + 2\mu & 0 \\ 0 & 0 & 2\mu \end{pmatrix} \tag{33}$$

5.2.2. Macroscopic Biot's tensor

Similarly to the computation of the homogenized stiffnesses, the macroscopic Biot's tensor α^H is obtained through the solution of problem (22), for $p^{(0)} = 1$ and $\epsilon^x \left(u^{(0)}\right) = 0$. As in the previous case, a parametric analysis showing the influence of the crack properties on the global, homogenized answer is proposed. The parametric analysis covers the range: $0 \leq \overrightarrow{A} \leq 1$, $\overrightarrow{A}$ being a dimensionless parameter. Fig.5 (left) illustrates a quasi-linear relationship between the homogenized Biot matrix α^H and the Biot vector $\overrightarrow{A}$ in the thin elastic layer. The difference of slopes between α_{11}^H and α_{22}^H in fig.5 (left) shows, as expected, that a dependence exists between the homogenized Biot matrix α^H and crack stiffness C, this is shown in fig. 5 (right).

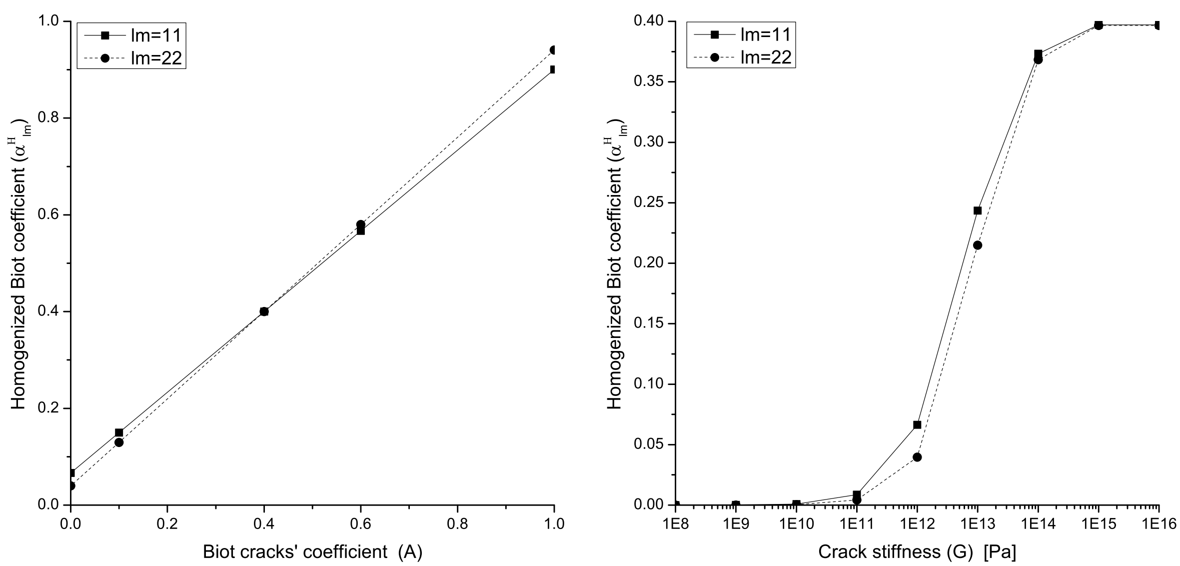

Figure 5: Left: relation between Biot coefficient in the cracks and homogenized Biot coefficient. Right: relation between Crack stiffness (G) and homogenized Biot coefficient for $\overrightarrow{A} = \overrightarrow{0}$ and $\overrightarrow{\alpha} = 0.4 \cdot \overrightarrow{I}$.

5.3. Numerical validation: permeability

Similarly to the computation of the macroscopic mechanical characteristics of the homogenized poroelastic medium and according to section 3.5, the macroscopic permeability matrix is obtained through the solution of problem (24), for a macroscopic pressure gradient $\overrightarrow{\mathrm{grad}}^x\, p^{(0)}$ successively parallel to the directions 1 and 2, i.e. respectively $\vec{e}_1$ and $\vec{e}_2$. For those computations, the permeability of the cracks is assumed homogeneous and equal to $K = 1 \cdot 10^{-7} m/s$ with a crack thickness equal to 0.001 times the cell size.

Due to the shape of the considered cell and the homogeneity of the permeabilities of the porous matrix and of the cracks, it can be seen that the solution $p^{(1)}$ of the problem (24) for $\overrightarrow{\mathrm{grad}}^x\, p^{(0)} = \vec{e}_1$ is constant and, according to (25c), gives:

$$k_{11}^H = k + \frac{K}{Y_2}$$

Consistently with the remark 1, k_{21}^H is found to be equal to 0. The computation of k_{22}^H needs the solution of problem (24) for $\overrightarrow{\text{grad}}^x\, p^{(0)} = \vec{e}_2$. Finally the macroscopic permeability matrix reads:

$$\underline{\underline{k}}^H = \begin{pmatrix} 2,986 \cdot 10^{-10} & 0 \\ 0 & 2,324 \cdot 10^{-10} \end{pmatrix}$$

6. Numerical results: damage

The purpose of this section is to - partially- study the macroscopic behaviour of the cracked porous medium when the cracks are damageable. As in the previous section, the macroscopic behaviour is obtained through the solution of a self balanced problem set on the elementary cell, namely problem (31). The self balanced filtration problem (24) remains essentially unchanged and is not considered in this section.

The main differences with the computations of section 5 are that the problem is not linear and that the data of problem (31) are the histories of the macroscopic strain and the pressure. A Newton's method and a time stepping procedure is then required.

The constitutive equation is numerically determined for two typical geomechanics examples, an oedometric and a biaxial tests in drained condition. Atmospheric pressure is neglected, that means that $p^{(0)} = 0$, so the medium is in fact dry.

The geometry of the cell as well the Young's modulus and the Poisson's ratio of the porous matrix and the elastic stiffness of the undamaged cracks are given in section 5.

6.1. Oedometric loading

Damage model is first applied in the case of an oedometric test. An arbitrary uniaxial macrostrain $\epsilon^x\left(u^{(0)}\right) = -0,01\vec{e}_1 \otimes \vec{e}_1$ is applied in 20 steps, length-like feature of the material of the cracks $D = 0.008$ times the size of the cell.

Elastic coefficients of the poroelastic medium:

$$\underline{\underline{c}}^H = \begin{pmatrix} 3,36 \cdot 10^9 & 1,44 \cdot 10^9 & 0 \\ 1,44 \cdot 10^9 & 3,36 \cdot 10^9 & 0 \\ 0 & 0 & 1,92 \cdot 10^9 \end{pmatrix} \tag{34}$$

And cracks:

$C_T = C_N = C = 10^{12}\text{Pa/m}$.

The stress-strain curve is given in fig. 7 as well as the number of iterations needed for convergence in the Newton Method, note that the peak of 38 iterations in the 16th time step corresponds to the localization of the strain in the crack network for an applied macrostrain around $\varepsilon_{11} = 6.4 \times 10^{-3}$.

6.2. Biaxial test loading

The second considered example corresponds to a biaxial test. The main difference with the oedometric test is that the loading is mixed in the sense that, roughly, it consists in a confining pressure and a gradually increasing longitudinal strain. However, the solution of the self balance problem (31) yields the stress in terms of the history of the strain and a suitable procedure, which goes as follows, is needed to simulate the biaxial test.

In the time stepping scheme, the finite element solution of the problem (31) yields the stress σ^n at the end of the step n in terms of the strain ϵ^n at the end of the step:

$$\sigma^n = \mathcal{T}^n\left(\epsilon^n\right)$$

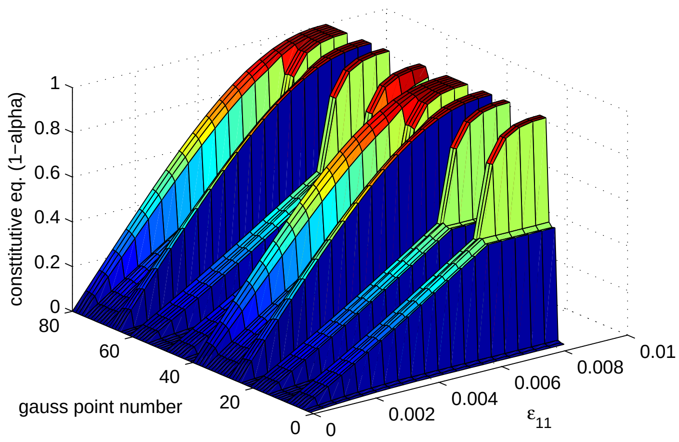

Figure 6: Oedometric test: evolution of the damage law for all the Gauss points of the cracks. D=0.008

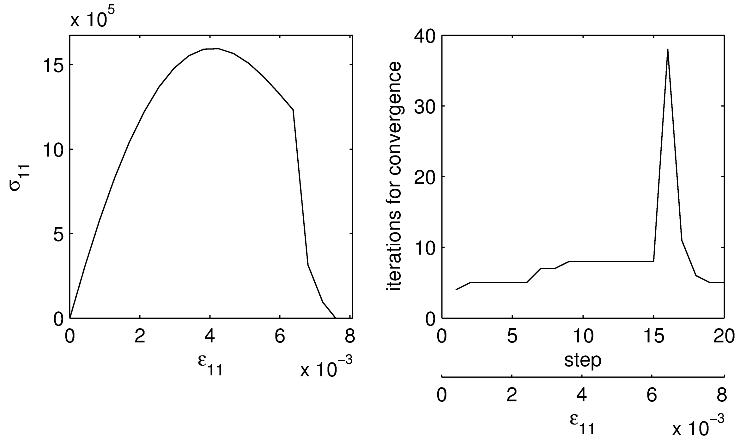

Figure 7: Oedometric test. Left: stress-strain 11 axis, right: number of iterations for convergence of Newton method.

Let's decompose the stress σ^n and strain ϵ^n on two supplementary subspaces E_1 and E_2 of the space of second order tensors:

$$\sigma^n = \sigma_1^n + \sigma_2^n$$
$$\epsilon^n = \epsilon_1^n + \epsilon_2^n$$

where σ_1^n and ϵ_2^n are given and σ_2^n and ϵ_1^n are unknown. The problem to be solved reads:

$$\begin{aligned} &\text{Given } \sigma_1^n \in E_1 \text{ and } \epsilon_2^n \in E_2, \text{ find } \sigma_2^n \in E_2 \text{ and } \epsilon_1^n \in E_1 \text{ such that :} \\ &\sigma_1^n + \sigma_2^n - \mathcal{T}^n \left(\epsilon_1^n + \epsilon_2^n\right) = 0 \end{aligned} \tag{35}$$

$\mathcal{T}^n$ is non-linear, so the Newton method is used to determine the solution of that problem. The linearized equation to be solved in $\delta\epsilon_1^{n(k)}$ and $\delta\sigma_2^{n(k)}$ at the iteration (k) of the Newton's method reads:

$$\delta\sigma_2^{n(k)} - C^n : \delta\epsilon_1^{n(k)} + \sigma_1^n + \sigma_2^{n(k)} - \mathcal{T}^n\left(\epsilon_1^{n(k)} + \epsilon_2^n\right) = 0$$

where $C^n = \frac{d\mathcal{T}^n}{\delta\epsilon^n}$ is computed at $\left(\epsilon_1^{n(k)} + \epsilon_2^n\right)$. Then the approximated solution at step k is updated into:

$$\begin{aligned}\epsilon_1^{n(k+1)} &= \epsilon_1^{n(k)} + \delta\epsilon_1^{n(k)} \\ \sigma_2^{n(k+1)} &= \sigma_2^{n(k)} + \delta\sigma_2^{n(k)}\end{aligned}$$

The procedure is then applied to the case of a biaxial test. As a first loading sequence, a confining pressure $p_c = 1 \cdot 10^7 Pa$ is gradually applied (time steps from 1 to 16), then a strain $\epsilon_{22} = 0.04$ (time steps from 17 to 40) is gradually imposed, the confining pressure remaining constant. Similarly to the previous section, the macroscale values $p_c = 1 \cdot 10^7$Pa and $\epsilon_{22} = 0.04$ are chosen in order to create a significant amount of damage.

So, in the first part of the loading, the time step n going from 1 to 16, the total stress σ^n is given ($\sigma_1^n = \sigma^n$, $\sigma_2^n \equiv 0$):

$$\underline{\underline{\sigma}}^n = \frac{n}{16}\begin{pmatrix} p_c & 0 \\ 0 & p_c \end{pmatrix}$$

$1 \leq n \leq 16$ and the whole strain ϵ^n is unknown ($\epsilon_1^n = \epsilon^n$, $\epsilon_2^n \equiv 0$).

In the second part of the loading, $17 \leq n \leq 40$, the data are:

$$\underline{\underline{\sigma}}_1^n = \begin{pmatrix} p_c & 0 \\ 0 & 0 \end{pmatrix} , \ \underline{\underline{\epsilon}}_2^n = \frac{n-16}{24}\begin{pmatrix} 0 & 0 \\ 0 & \epsilon_{22} \end{pmatrix}$$

and the unknowns are:

$$\underline{\underline{\sigma}}_2^n = \begin{pmatrix} 0 & 0 \\ 0 & \sigma_{22}^n \end{pmatrix} , \ \underline{\underline{\epsilon}}_1^n = \begin{pmatrix} \epsilon_{11}^n & \epsilon_{12}^n \\ \epsilon_{12}^n & 0 \end{pmatrix}$$

Results are given in fig.8.

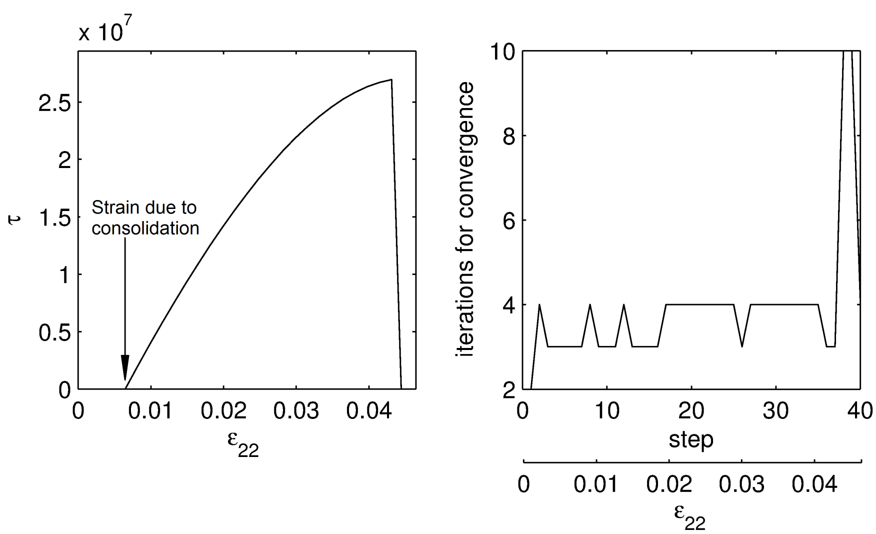

Figure 8: Biaxial compression test. Left: stress-strain 22 axis, right: number of iterations for convergence of Newton method.

6.3. Oedometric and biaxial tests: discussion of the results

Microscale damage implies softening of the homogenised stress tensor. In accordance with the combination of the applied macrostrains on the time-history, cracks are affected in a different manner and, consequently, the stress tensor evolves depending on these macrostrain paths.

As in the example given in fig.7, the homogenized stresses σ^H_{1111}, σ^H_{1122} and σ^H_{1212} are similarly affected but σ^H_{2222} has significantly less degradation due to the non symmetric loading. Accordingly to the lemma of AppendixC the stresses σ^H_{1112} and σ^H_{2212} and their symmetric σ^H_{1211} and σ^H_{1222} are equal to zero.

It can also be observed that for a large strain, which also implies high damage, some homogenized stresses become zero. In other terms, the sample can be considered as broken after a given iteration. Moreover, the damaged (but not completely broken) cracks close, leading to a non unique solution [12]. An example of this (deformed configuration and Von Mises stress field) can be seen in fig.9.

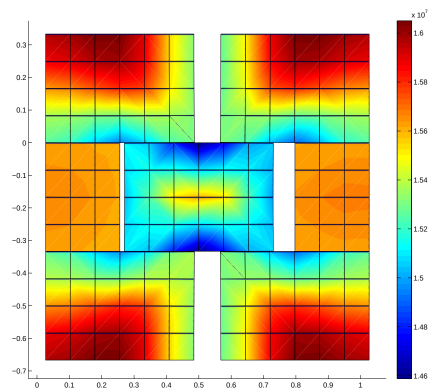

Figure 9: Loss of solution uniqueness (deformed configuration and Von Mises stress field)

The loss of uniqueness of the solution, also known as bifurcation, can be understood in numerical modeling as the equivalent of strain localization in experimental studies. This point is discussed in the following section.

Broken joints make the stiffness matrix of the FEM model close to singularity. To avoid numerical problems, a modified damage evolution law preserving 1% of stiffness for a totally broken crack has also been used, though leading to results similar to the previous ones as well as to bifurcation.

6.4. Bifurcation and loss of uniqueness

Uniqueness can not be assured in problems presenting softening (e.g. [12]). It can be proved that when the stiffness decreases, the discharge branch can have any slope, and the solution is not known a priori. From the point of view of PDE formulation, when some of the eigenvalues of the stiffness tensor become zero, the problem loses its ellipticity in those points, so any strain applied in the direction of the vanishing eigenvalues will fulfill the equilibrium condition, thus making the solution not unique.

The loss of uniqueness, or bifurcation, in numerical models is an interesting phenomenon that agrees with experimental observations; nevertheless, it represents a numerical challenge, i.e. decrease of efficiency of Newton method, some of the solutions may not be found, representing a main con- cern in e.g. reliability or failure analysis.

In this section some different possible solutions for a biaxial compression test are explored. This attempt should not be considered as an exhaustive procedure to find all the possible solutions but rather a demonstration of the existence of bifurcation.

The same problems presented in the previous sections are solved using different number of time steps, i.e. 29 and 30 and the different evolution of the tangent elastic moduli is given in fig. 10.

The asimptotic result of the homogenized stress σ^H_{2222} for a strain $\epsilon_{11} = 0.85\%$ is compared in a parametric study on the number of time steps and parameter D (Fig. 11), 3600 computations are performed, being the results polarized in two regions, one around $\sigma^H_{2222} = 2.1\text{x}10^8$Pa and another around $\sigma^H_{2222} = 2.6\text{x}10^8$Pa, this evidences that two different localization modes of the crack network represent two feasible solutions of the problem.

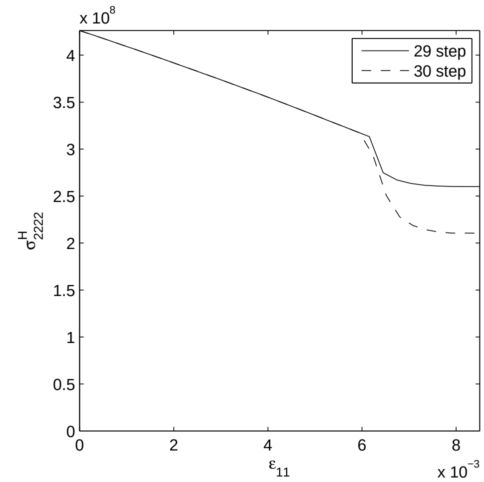

Figure 10: Evolution of the tangent elastic moduli for different time-history steps in an oedometric test.

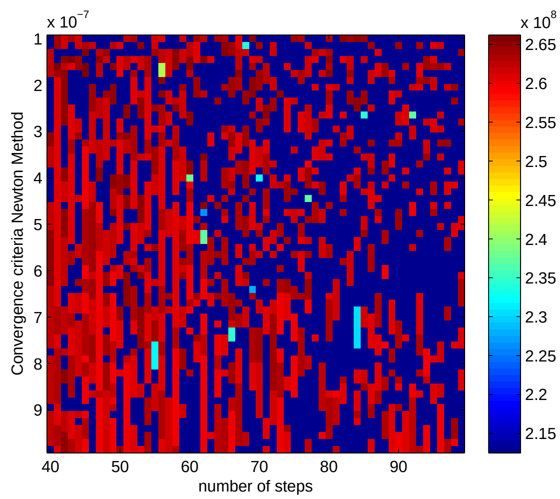

Figure 11: Bifurcation study: value of , σ^H_{22} at the end of an oedometric loading, parametric study changin number of steps and parameter D.

7. Conclusions

A theoretical and numerical approach for describing the macroscopic poroelastic properties of a saturated, deformable, cracked porous medium has been presented in this paper.

The first part of the work is devoted to the formulation of the equations describing the hydro-mechanical behavior of the cracked fully-saturated porous medium. Under the classical assumption of well-separated scales, the asymptotic homogenization method has allowed to obtain the macroscopic description of the whole porous medium. The homogenized coefficients (elastic tensor, Biot's coefficient and modulus, permeability tensor) embody all the information about the microstructure by means of the characteristic functions that describe the small oscillations of the primary variables of the problem. The numerical solution of the problem over the elementary cell has allowed to obtain the homogenized coefficients in the linear case. The macroscale reflects the anisotropy coming from the microscale configuration. The damage associated to the cracks' opening implies a degradation of the homogenized stresses and results in a non-linear problem requiring a Newton-like procedure. The methodology has been applied to two different cases, a strain controlled path, i.e. an oedometric test and a biaxial test have been explored. This latter problem requires the development of a controllability scheme so that any time-history of stress-strain can be applied. At some point of the time-history (as expected for a mechanical damage problem), for high enough values of damage, bifurcation is observed, leading to a quick degradation of the homogenized stresses and to a loss of solution uniqueness. This latter aspect has been discussed but future works should explore other aspects such as the influence of the cell size. A larger cell means a lower constriction to the possible failure mechanisms, i.e. a lower strain value for the sudden quick degradation triggering is expected. Moreover, the obtained homogenized stresses should be used in real-scale geomechanics or engineering problems. Therefore, further research including the calibration of the microscale coefficients with real materials should be performed.

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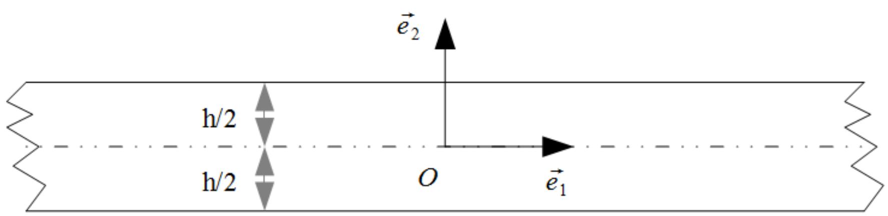
AppendixA. Very thin layer of a deformable porous medium

The purpose of this appendix is to heuristically justify the poroelastic modeling (2) of a very thin layer of poroelastic medium of weak stiffness and large permeability. A more rigorous approach for a partly similar but simpler case can be found in Caillerie [9].

Let's consider a very thin 2D deformable porous medium of thickness h in the x_2 direction governed by the Biot's equations. This thin layer is embedded in another deformable porous medium the poroelastic characteristics do not depend on the thickness of the thin layer.



The porosity of this medium is assumed to be close to 1 so that its elastic coefficients are very small and its permeability very high. To make that precise, we consider the equations (1) where the the elastic stiffness tensor c^h, the Biot's tensor and coefficient α^h and β^h and the permeability k^h depend on h in the following manner:

$$c^h = h\tilde{C} \ ; \ \alpha^h = \alpha \ (\alpha^h \text{ is constant}) \ ; \ \beta^h = \frac{\beta}{h} \ ; \ k^h = \frac{K}{h}$$

Taking into account the symmetries $\tilde{C}_{ijkl} = \tilde{C}_{ijlk}$ and $\alpha_{ij} = \alpha_{ji}$, the Biot's equation can be rewritten, using the index notation:

$$\frac{\partial \sigma_{ij}}{\partial x_j} = 0 \tag{A.1a}$$

$$\sigma_{ij} = h\tilde{C}_{ijkl}\frac{\partial u_k}{\partial x_l} - p\alpha_{ij} \tag{A.1b}$$

$$\kappa = \alpha_{ij}\frac{\partial u_i}{\partial x_j} + \frac{\beta}{h}p \tag{A.1c}$$

$$\frac{\partial q_i}{\partial x_i} + \dot{\kappa} = 0 \tag{A.1d}$$

$$q_i = -\frac{K}{h}\frac{\partial p}{\partial x_i} \tag{A.1e}$$

Due to the continuity conditions on the two sides of the layer, it is consistent to assume that the displacement and the pressure in the layer are of order 1 with respect to h:

$$\vec{u} = O\left(1\right) \tag{A.2a}$$

$$p = O\left(1\right) \tag{A.2b}$$

Consistently with the equations (A.1), the stress tensor σ and the relative flow of fluid $\vec{q}$ depend on h as :

$$\sigma = O(1) \tag{A.3a}$$

$$q_1 = O(h^{-1}) \tag{A.3b}$$

$$q_2 = O(1) \tag{A.3c}$$

moreover, according to (A.1c), the variation of porosity depends on h as:

$$\kappa = 0\left(h^{-1}\right) \tag{A.4}$$

The integration of the balance equation (A.1a) in x_2 over $\left[-\frac{h}{2}, \frac{h}{2}\right]$ yields:

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\partial \sigma_{i1}}{\partial x_1}\mathrm{d}x_2 + [[\sigma_{i2}]] = 0 \tag{A.5}$$

where the jump $[[f]]$ of a function f is:

$$[[f]] = f\left(\frac{h}{2}\right) - f\left(-\frac{h}{2}\right)$$

as $\sigma = O\left(1\right)$, for $h \to 0$, the equation (A.5) reads:

$$[[\sigma_{i2}]] = 0$$

Which means that σ_{i2} is continuous.

The integration of (A.1c) yields:

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \kappa\,\mathrm{d}x_2 = \alpha_{i1}\int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\partial u_i}{\partial x_1}\mathrm{d}x_2 + \alpha_{i2}\,[[u_i]] + \frac{\beta}{h}\int_{-\frac{h}{2}}^{\frac{h}{2}} p\,\mathrm{d}x_2$$

which, taking into account the order of magnitude of p and κ with respect to h (see (A.2a) and (A.4)), yields in the limit $h \to 0$:

$$\kappa^c = A_i\,[[u_i]] + \beta p$$

where the components of the vector $\vec{A}$ are $A_i = \lim_{h\to 0} \frac{1}{h} \int_{-\frac{h}{2}}^{\frac{h}{2}} \alpha_{ij}\,\mathrm{d}x_2$ and:

$$\kappa^c = \lim_{h\to 0} \int_{-\frac{h}{2}}^{\frac{h}{2}} \kappa\,\mathrm{d}x_2$$

In the same way, the integration of (A.1d) yields:

$$\frac{d}{dx_1} \int_{-\frac{h}{2}}^{\frac{h}{2}} q_1 \mathrm{d}x_2 + [[q_2]] + \int_{-\frac{h}{2}}^{\frac{h}{2}} \dot{\kappa}\,\mathrm{d}x_2 = 0$$

according to the order of magnitude of q_1, q_2 and κ (see (A.3) and (A.4)), for $h \to 0$ that reads:

$$\frac{d}{dx_1} Q + [[q_2]] + \dot{\kappa}^c = 0$$

where η is assumed independent of x_2 and where:

$$Q = \lim_{h\to 0} \int_{-\frac{h}{2}}^{\frac{h}{2}} q_1 \mathrm{d}x_2$$

The integration of the constitutive equation (A.1b) yields:

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{ij}\,\mathrm{d}x_2 = h\tilde{C}_{ijk1} \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\partial u_k}{\partial x_1} \mathrm{d}x_2 + h\tilde{C}_{ijk2}\,[[u_k]] - \int_{-\frac{h}{2}}^{\frac{h}{2}} p\alpha_{ij}\,\mathrm{d}x_2$$

Passing to the limit $h \to 0$ after dividing by h, that yields:

$$\sigma_{i2} = C_{ik}\,[[u_k]] - pA_i$$

where the components of the matrix of the linear application C and of the vector $\vec{A}$ are respectively $C_{ij} = \tilde{C}_{i2j2}$ and $A_i = \lim_{h\to 0} \frac{1}{h} \int_{-\frac{h}{2}}^{\frac{h}{2}} \alpha_{i2}\,\mathrm{d}x_2$. It can be seen that according to the usual symmetries of elastic stiffnesses the crack stifness tensor C is symmetrical.

Remark 2. If the material of the crack is isotropic then $\tilde{C}_{1222} = 0$ and $C_{12} = C_{21} = 0$. In the same way, if the Biot's tensor α of the material of the crack is proportional to the identity $\mathbb{I}$ then $A_1 = 0$, that is to say that $\vec{A}$ is normal to the crack.

The integration of the Darcy's law (A.1b) yields:

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} q_1\,\mathrm{d}x_2 = -K\frac{1}{h} \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\partial p}{\partial x_1} \mathrm{d}x_2$$

that yields, passing to the limit $h \to 0$:

$$Q = -K\frac{\partial p}{\partial x_1}$$

It can be noticed that, in the considered case, $\sum_i \sigma_{i2}\vec{e}_i = \sigma \centerdot \vec{n} = \vec{T}$ and $q_2 = \vec{q} \centerdot \vec{n}$, where $\vec{n} = \vec{e}_2$ is the normal to the thin layer, and moreover that x_1 is the curvilinear coordinate s along the crack so the equations along the crack can be written:

$$\vec{T} = C \centerdot [[\vec{u}]] - p\vec{A}$$
$$\kappa^c = \vec{A}.\,[[\vec{u}]] + \beta p$$
$$\frac{dQ}{ds} + [[\vec{q}]]\,.\vec{n} + \dot{\kappa}^c = 0$$
$$Q = -K\frac{dp}{ds}$$

which are the interface conditions to be considered along any crack even not a straight one. Moreover, if the material of the crack is isotropic and its Biot's tensor α is proportional to the identity then the stiffness tensor of the crack and its Biot's vector read:

$$C = C_T \vec{\tau} \otimes \vec{\tau} + C_N \vec{n} \otimes \vec{n} \tag{A.6}$$

$$\vec{A} = A\vec{n} \tag{A.7}$$

AppendixB. Proof of the damage evolution law (30) at order zero

According to (11d), the expansion of $\| \left[\left[\vec{u}^{(e)}\right]\right] \|$ reads:

$$\left\| \left[\left[\vec{u}^{(e)}\right]\right] \right\| = e \left\| \left[\left[\vec{u}^{(1)}\right]\right] \right\| + e^2 \left(\cdots\right)$$

therefore, that of $f\left(\frac{\| [[\vec{u}^{(e)}]] \|}{D^{(e)}}\right)$ reads:

$$f\left(\frac{\| [[\vec{u}^{(e)}]] \|}{D^{(e)}}\right) = f\left(\frac{\| [[\vec{u}^{(1)}]] \|}{D}\right) + e^2 \left(\cdots\right)$$

Using that expansion, the evolution law (28) entails that:

$$\forall e, t, \tau, 0 \leq \tau \leq t\,,\; d^{(0)}\left(t\right) + e d^{(1)}\left(t\right) + e^2 d^{(2)}\left(t\right) + \cdots \geq f\left(\frac{\| [[\vec{u}^{(1)}]]\left(\tau\right) \|}{D}\right) + e^2 \left(\cdots\right)$$

then necessarily:

$$\forall t, \tau, 0 \leq \tau \leq t\,,\; d^{(0)}\left(t\right) \geq f\left(\frac{\| [[\vec{u}^{(1)}]]\left(\tau\right) \|}{D}\right) \tag{B.1}$$

otherwise for any small enough e, $d^{(e)}\left(t\right)$ would be less than $f\left(\frac{\| [[\vec{u}^{(e)}]](\tau) \|}{D}\right)$ for some τ comprised between 0 and t. The equation (B.1) is clearly equivalent to:

$$\forall t \geq 0\,,\; d^{(0)}\left(t\right) \geq \sup_{0\leq\tau\leq t} f\left(\frac{\| [[\vec{u}^{(1)}]]\left(\tau\right) \|}{D}\right)$$

Moreover, if for some $\tau, 0 \leq \tau \leq t$, $d^{(0)}\left(t\right)$ is such that:

$$d^{(0)}\left(t\right) > f\left(\frac{\| [[\vec{u}^{(1)}]]\left(\tau\right) \|}{D}\right)$$

then for small enough e's we have:

$$d^{(e)}\left(t\right) > f\left(\frac{\| [[\vec{u}^{(e)}]]\left(\tau\right) \|}{D^{(e)}}\right) \tag{B.2}$$

That means that $d^{(0)}\left(t\right) > \sup_{0\leq\tau\leq t} f\left(\frac{\| [[\vec{u}^{(1)}]](\tau) \|}{D}\right)$ entails that, or small enough $e's$:

$$d^{(e)}\left(t\right) > \sup_{0\leq\tau\leq t} f\left(\frac{\| [[\vec{u}^{(e)}]]\left(\tau\right) \|}{D^{(e)}}\right)$$

which is not possible for $d^{(e)}\left(t\right) = \sup_{0\leq\tau\leq t} f\left(\frac{\| [[\vec{u}^{(e)}]](\tau) \|}{D^{(e)}}\right)$, consequently $d^{(0)}\left(t\right)$ is exactly the maximum of $f\left(\frac{\| [[\vec{u}^{(1)}]](\tau) \|}{D}\right)$:

$$\forall t \geq 0\,,\; d^{(0)}\left(t\right) = \sup_{0\leq\tau\leq t} f\left(\frac{\| [[\vec{u}^{(1)}]]\left(\tau\right) \|}{D}\right)$$

which is the evolution law (30).

AppendixC. Symmetries

Let $\vec{u}\,(\tau)$, defined and periodic on Y, and $\Sigma\,(\tau)$ belonging to $\mathcal{L}^S$ (space of the symmetric second order tensors), $0 \leq \tau \leq t$, be the solution of the problem:

$$
\begin{aligned}
&\text{Given the histories } \{E\,(\tau)\,;\,0 \leq \tau \leq t\} \text{ and } \{P\,(\tau)\,;\,0 \leq \tau \leq t\}\,,\\
&\qquad\text{find } \vec{u}\,(\vec{x},\vec{y},\tau)\,,\, d\,(\vec{x},\vec{y},\tau)\,,\, Y\text{-periodic and } \Sigma\,(\tau)\,, \tau \in [0,t]\,, \text{ such that:}\\
&\qquad\forall \tau \in [0,t]\, \forall \vec{v}, Y\text{-periodic}, \forall E^* \in \mathcal{L}^S\,,\ \int_Y \left(c : \left(E\,(\tau) + \epsilon^y\,(\vec{u}\,(\tau))\right) - P\,(\tau)\,\alpha\right) : \epsilon^y\,(\vec{v})\ \mathrm{d}s_y\\
&\qquad\qquad + \int_{\Gamma^Y} \left((1 - d\,(\tau))\, C \,.\, [[\vec{u}\,(\tau)]] - P\,(\tau)\,\vec{A}\right) . \,[[\vec{v}]]\ \mathrm{d}l_y - |Y|\,\Sigma\,(\tau) : E^* = 0\\
&\qquad\qquad\qquad \text{with } d\,(\tau) = \sup_{0 \leq \rho \leq \tau} f\left(\frac{\|\,[[\vec{u}\,(\rho)]]\,\|}{D}\right)
\end{aligned}
\tag{C.1}
$$

where $\forall \tau\,,\ E\,(\tau) \in \mathcal{L}^S$ and $P\,(\tau) \in \mathbb{R}$.

Taking $E^* = 0$, $E\,(\tau) = \epsilon^x\left(\vec{u}^{(0)}\,(\tau)\right)$ and $P\,(\tau) = p^{(0)}\,(\tau)$, the previous problem comes down to (31). Posing $\sigma = c : \epsilon^y\,(E\,(\tau)\,.\vec{y} + \vec{u}\,(\tau)) - P\alpha$ and taking $\vec{v} = 0$ yield $\Sigma\,(\tau) = \langle \sigma\,(\tau) \rangle = \frac{1}{|Y|}\int_Y \sigma\,(\tau)\ \mathrm{d}s_y$.

Now, let's assume that there exists an isometry-valued function $\tau \to R\,(\tau)$ defined in $[0,t]$ $(R^{-1}\,(\tau) = R^T\,(\tau))$ such that, for all τ, $R\,(\tau)$ leaves the cell Y and the cracks Γ^Y unchanged and such that the tensors c, α, C and the vector $\vec{A}$ satisfy:

$$
\forall M, N \in \mathcal{L}^S\,,\ \left(R\,(\tau) \circ M \circ R^T\,(\tau)\right) : \left(c\,(R\,(\tau)\,.\vec{y}) : \left(R\,(\tau) \circ N \circ R^T\,(\tau)\right)\right) = M : (c\,(\vec{y}) : N) \tag{C.2a}
$$

$$
\left(R^T\,(\tau) \circ \alpha\,(R\,(\tau)\,.\vec{y}) \circ R\,(\tau)\right) = \alpha\,(\vec{y}) \tag{C.2b}
$$

$$
\left(R^T\,(\tau) \circ C\,(R\,(\tau)\,.\vec{y}) \circ R\right) = C\,(\vec{y}) \tag{C.2c}
$$

$$
\forall \vec{y} \in Y\,,\ R^T\,(\tau)\,.\vec{A}\,(R\,(\tau)\,.\vec{y}) = \vec{A}\,(\vec{y}) \tag{C.2d}
$$

Under those conditions, we have the following lemma:

Lemma. *Let $\vec{u}\,(\tau)$ and $\Sigma\,(\tau)$ be the solution of (C.1) for the data $E\,(\tau)$ and $P\,(\tau)$, $0 \leq \tau \leq t$ then $R\,(\tau)\,.\vec{u}\,(\tau)$ and $R^T\,(\tau) \circ \Sigma\,(\tau) \circ R\,(\tau)$ are solution of (C.1) for the data $R\,(\tau) \circ E \circ R^T\,(\tau)$ and $P\,(\tau)$.*

Proof. All the following algebraic calculi being performed for any time τ and the sake of simplicity, the variable τ is omitted.

Taking into account the relations (C.2), (C.1) reads:

$$
\begin{aligned}
\forall \vec{v}, Y\text{-periodic}, \forall E^* \in \mathcal{L}^S\,,\ &\int_Y \left[c\,(R.\vec{y}) : \left(R \circ \epsilon^y\,(E.\vec{y} + \vec{u}) \circ R^T\right)\right] : \left(R \circ \epsilon^y\,(E^*.\vec{y} + \vec{v}) \circ R^T\right)\ \mathrm{d}s_y\\
&- \int_Y P\left(R^T \circ \alpha\,(R.\vec{y}) \circ R\right) : \left(\epsilon^y\,(E^*.\vec{y} + \vec{v}) \circ\right)\ \mathrm{d}s_y\\
&+ \int_{\Gamma^Y} \left(\left(R^T \circ C\,(R.\vec{y}) \circ R\right) . \,[[\vec{u}]] - P R^T.\vec{A}\,(R.\vec{y})\right) . \,[[\vec{v}]]\ \mathrm{d}l_y - |Y|\,\Sigma : E^* = 0
\end{aligned}
$$

that is too:

$$
\begin{aligned}
\forall \vec{v}, Y\text{-periodic}, \forall E^* \in \mathcal{L}^S\,,\ &\int_Y \left[c\,(R.\vec{y}) : \left(\epsilon^y\,(R \circ E.\vec{y} + R.\vec{u}) \circ R^T\right)\right] : \left(\epsilon^y\,(R \circ E^*.\vec{y} + R.\vec{v}) \circ R^T\right)\ \mathrm{d}s_y\\
&- \int_Y P\alpha\,(R.\vec{y}) : \left(\epsilon^y\,(R \circ E^*.\vec{y} + R.\vec{v}) \circ R^T\right)\ \mathrm{d}s_y\\
&+ \int_{\Gamma^Y} \left((C\,(R.\vec{y}))\,.\,(R.\,[[\vec{u}]]) - P\vec{A}\,(R.\vec{y})\right) . \,(R.\,[[\vec{v}]])\ \mathrm{d}l_y - |Y|\,\Sigma : E^* = 0
\end{aligned}
$$

In the change of variables $\vec{y} \leftrightarrow \vec{\tilde{y}} = R.\vec{y}$, the cell Y and the cracks Γ^Y remained unchanged. Moreover, $\nabla^y \vec{v} = \nabla^{\tilde{y}} \vec{v} \circ R$ and, as R is an isometry, $\mathrm{d}s_{\tilde{y}} = \mathrm{d}s_y$ and $\mathrm{d}l_{\tilde{y}} = \mathrm{d}l_y$ on Γ^Y. So, by this change of variables, the previous formulation reads:

$$\forall \vec{v}, Y\text{-periodic}, \forall E^* \in \mathcal{L}^S\,,$$

$$\int_Y \left(c\left(\vec{\tilde{y}}\right) : \left(\epsilon^{\tilde{y}}\left(R\circ E\circ R^T.\vec{\tilde{y}} + R.\vec{u}\right)\right) - P\,\alpha\left(\vec{\tilde{y}}\right)\right) : \left(\epsilon^{\tilde{y}}\left(R\circ E^*\circ R^T.\vec{\tilde{y}} + R.\vec{v}\right)\right)\,\mathrm{d}s_{\tilde{y}}$$
$$+\int_{\Gamma^Y}\left(\left(C\left(\vec{\tilde{y}}\right)\right)\centerdot\left(R.\left[\left[\vec{u}\right]\right]\right) - P\vec{A}\left(\vec{\tilde{y}}\right)\right)\centerdot\left(R.\left[\left[\vec{v}\right]\right]\right)\,\mathrm{d}l_{\tilde{y}} - |Y|\,\Sigma : E^* = 0$$

that is too, posing $\vec{\tilde{v}} = R.\vec{v}$ and $\tilde{E}^* = R\circ E^*\circ R^T$:

$$\forall \vec{\tilde{v}}, Y\text{-periodic}, \forall \tilde{E}^* \in \mathcal{L}^S\,,\ \int_Y \left(c\left(\vec{\tilde{y}}\right) : \left(\epsilon^{\tilde{y}}\left(R\circ E\circ R^T.\vec{\tilde{y}} + R.\vec{u}\right)\right) - P\,\alpha\left(\vec{\tilde{y}}\right)\right) : \left(\epsilon^{\tilde{y}}\left(\tilde{E}^*.\vec{\tilde{y}} + \vec{\tilde{v}}\right)\right)\,\mathrm{d}s_{\tilde{y}}$$
$$+\int_{\Gamma^Y}\left(\left(C\left(\vec{\tilde{y}}\right)\right)\centerdot\left(R.\left[\left[\vec{u}\right]\right]\right) - P\vec{A}\left(\vec{\tilde{y}}\right)\right)\centerdot\left(\left[\left[\vec{\tilde{v}}\right]\right]\right)\,\mathrm{d}l_{\tilde{y}} - |Y|\left(R^T\circ\Sigma\circ R\right) : \tilde{E}^* = 0$$

The comparison of that formulation with (C.1) shows that $R\left(\tau\right).\vec{u}$ and $R^T\left(\tau\right)\circ\Sigma\circ R\left(\tau\right)$ are the solution of the problem (C.1) for the data $R\left(\tau\right)\circ E\circ R^T\left(\tau\right)$ and P. $\square$